HAMILTONIAN FORMALISM FOR GENERAL-RELATIVISTIC ADIABATIC FLUIDS

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We derive the Hamiltonian structures of three theories: non-relativistic, special-relativistic, and general-relativistic adiabatic fluids, each in the Eulerian representation in Riemannian space (or Lorentzian spacetime), all by the same procedure using standard variational principles. The evolution in each case is generated by a Hamiltonian that is equivalent to that obtained from a canonical analysis. For the gravitational variables, the Poisson bracket has the usual canonical symplectic structure. However, for the fluid variables, the three theories all share the same Lie–Poisson bracket, when expressed in the appropriate spaces of physical variables constructed here. This shared Lie–Poisson bracket is associated to the dual of the semidirect-product Lie algebra of vector fields acting on differential forms. An immediate consequence of this shared structure is that each of these theories possesses an infinite family of conservation laws: the so-called "Casimirs" that belong to the kernel of the Lie–Poisson bracket. The role of these Casimirs in the study of Lyapunov stability (or dynamic stability) for fluid equilibria is discussed. The relationship of this approach to other approaches in the literature is also discussed.

1. Introduction

This paper *constructs* the noncanonical Hamiltonian structure for general relativistic adiabatic fluids, by starting from a physically motivated action principle and using standard variational techniques.

In general-relativistic (GR) theories, Hamiltonian methods and initial-value procedures can be applied once the theory has been translated into $3 + 1$ (actually, $n + 1$) language, by using the so-called ADM decomposition due to Arnowitt, Deser and Misner [1]. The ADM formalism is reviewed in Fischer and Marsden [2], Isenberg [3], Isenberg and Nester [4], Isham [5], and Kuchar [6]. This approach starts from an action principle for the gravitational field and other fields, and leads to a Hamiltonian formalism,

$$\partial_t F = \{ H, F \}_c,$$

for the dynamical evolution, where $\{ , \}_c$ is the canonical (symplectic) Poisson bracket, H is the Hamiltonian, and F is any functional on the space of (canonically conjugate) dynamical variables. The ADM formalism has proven useful in the study of theoretical and numerical problems in the construction of space–time solutions from initial data, see, e.g., Isenberg [3] and York [7].

Arnowitt, Deser and Misner [1] (hereafter abbreviated ADM) showed that the vacuum Einstein equations are Hamiltonian with a symplectic Poisson bracket in terms of canonically conjugate gravitational fields g_{ij} and π^{ij}, with $i, j = 1, 2, 3$. (See Fischer and Marsden [2] for a more extensive discussion of the Hamiltonian properties of the vacuum Einstein equations.) The ADM Hamiltonian density is a linear combination (with, in general, time-dependent coefficients) of the well-known GR constraints. These constraints are handled by properly choosing the initial data, since the GR constraints are nondynamical: if they hold initially, they will continue to hold. The time-dependent coefficients in the Hamiltonian density are the ADM lapse and shift functions, which must be prescribed for all time along with the initial data for each space–time solution.

Künzle and Nester [8] (hereafter abbreviated as KN) showed via the ADM approach that the equations for general relativistic adiabatic fluids (GRAF) are Hamiltonian with a symplectic Poisson bracket, when the fluid variables are represented using Lagrangian coordinates regarded as maps of spatial points (what we will call Eulerian independent variables) to reference points (called Lagrangian independent variables). For the Lagrangian representation of GRAF, ADM techniques have also been applied recently by Moncrief [9] using the variational principle due to Taub [10]. A covariant Lagrangian representation of GRAF appears in Tulczyjew [11].

Bao, Marsden and Walton [12] (hereafter abbreviated BMW) found heuristically that a noncanonical Poisson bracket involving a "Lie–Poisson" bracket for the fluid variables also exists for GRAF expressed in the $(3 + 1)$ language of the ADM formalism. A natural question, of course, is how this newly-discovered Lie–Poisson bracket should be related to the symplectic Poisson bracket obtained for GRAF in KN via the ADM approach. BMW suggest various ways of obtaining their Lie–Poisson bracket for GRAF from fluid descriptions in Lagrangian coordinates via the method of reduction. In particular, they suggest that there exists a map from the symplectic Poisson bracket in KN to their Lie–Poisson bracket, via reduction with respect to a certain (right) action of the group of diffeomorphisms of Lagrangian reference configurations, lifted to Lagrangian phase space. A different map is explored in the present work.

The present work constructs an explicit map from the symplectic Poisson bracket (obtained for GRAF in KN via the ADM approach using a fluid description in which Lagrangian coordinates are treated as Eulerian fields) to the Lie–Poisson bracket found heuristically in BMW. Elsewhere, Holm, Marsden and Ratiu [64] describe how this map fits into the mathematical framework of Marsden, Ratiu and Weinstein [14] and compare this map to the one suggested in BMW and discussed for nonrelativistic fluids in Holm,

Kupershmidt and Levermore [15], Marsden et al. [16], and Marsden, Ratiu, and Weinstein [14, 17].

More specifically, this paper systematically derives the Hamiltonian structures of nonrelativistic (NR), special relativistic (SR), and general relativistic adiabatic fluids (GRAF), each in the Eulerian representation, all by the same procedure using standard variational techniques as in ADM. The first objective of such a unified treatment is to keep the similarities, differences, and limiting processes among these three levels of description apparent at every stage. Another objective is to provide an explicit basis for: (1) extensions, such as seeking Hamiltonian structures for general relativistic systems including additional physics, such as magnetohydrodynamics, electromagnetic interactions, and Yang–Mills interactions; and (2) "technology transfer," such as the use in general relativistic systems of Hamiltonian methods for studying Lyapunov stability as in Holm, Marsden, Ratiu and Weinstein [13], or for developing approximation schemes using techniques from other fields, such as plasma physics, e.g., Whitham-averaged action principles as in Similon, Kaufman and Holm [18], but with the averaging done over the phase of high-frequency gravitational waves, instead of the phase of electromagnetic waves.

As for the first objective, the main similarity is that the resulting Hamiltonian structures for the fluid variables in NRAF, SRAF, and GRAF all share a *common* Lie–Poisson bracket when expressed in the appropriate space of variables constructed here. The NRAF Lie–Poisson bracket is due to Iwinski and Turski [19], although it was rediscovered by Dzyaloshinsky and Volovick [20] and by Morrison and Greene [21] (who also treated NR magnetohydrodynamics). The SRAF Lie–Poisson bracket is due to Bialynicki-Birula and Iwinski [22] for free-streaming pressureless fluids, and to Iwinski and Turski [19] for SRAF including pressure forces and electromagnetic interactions. The GRAF Lie–Poisson bracket is due to BMW. The Poisson structure shared by all three of these adiabatic fluids is the Lie–Poisson bracket shown in Holm and Kupershmidt [23] to be asso-

ciated to the dual of the semidirect-product Lie algebra of vector fields acting on differential forms. (This type of Lie–Poisson bracket is given a group theoretical interpretation in Marsden, Ratiu and Weinstein [14, 17].) An immediate consequence of this shared structure is that each of these theories possesses an infinite family of conservation laws: the so-called "Casimirs" that belong to the kernel of the Lie–Poisson bracket. The role of these Casimirs in the study of Lyapunov stability for fluid equilibria is discussed at the end of this paper.

There are two main differences between the Hamiltonian structures resulting for special relativistic and general relativistic fluid descriptions. First, the gravitational fields g_{ij} and π^{ij} appear in the Poisson bracket as a canonically conjugate pair, just as they do in the vacuum Einstein case. Second, the GR Hamiltonian density has the same constraint form as in the vacuum ADM case; although it does limit to the SRAF Hamiltonian density appropriately when g_{ij} is time independent, the ADM lapse is unity, the ADM shift vanishes, and then the gravitational coupling constant, $k = 16\pi G$ (G is Newton's constant), tends to zero. Thence, as c^{-2} tends to zero (c is the speed of light) the Hamiltonian density for SRAF tends to that for NRAF. We have made the dependence on k and c explicit everywhere in order that these limits are apparent. There also exist the much more subtle post-Newtonian "limits" (actually, approximations) neglected here, but treated by KN and by Futamase and Schutz [24]. The constraint form of the GRAF Hamiltonian introduces subtleties into the Hamiltonian formalism and its relation to the initial value problem that are not discussed in the present work. For comprehensive discussions of constraints in the GR initial value problem and reviews of the literature on this subject, see Isenberg [3] and York [7].

Besides the ADM approach, covariant action principles and Hamiltonian formulations for GRAF are also available in the literature. Covariant action principles for GRAF in the Lagrangian representation are given in Taub [10,

25–27], Kijowski and Tulczyjew [28], and Tulczyjew [11]. Such action principles for GRAF in the Eulerian representation are obtained by Ray [29, 30], Schutz [31, 32], and Schutz and Sorkin [33]. These authors use a covariant extension of the traditional Clebsch technique, which for non-relativistic fluids is given a final form in Seliger and Whitham [34]. Additional references for GRAF action principles appear in Misner, Thorne and Wheeler [35] (hereafter abbreviated as MTW). The covariant Clebsch method led Schutz [31, 32] to propose a symplectic Hamiltonian structure for GRAF based on using additional, nonphysical, Clebsch potentials. Kentwell [36] presents a covariant Lie–Poisson bracket for GRAF in a *given* space–time via the Clebsch map approach of Holm and Kupershmidt [23], starting from the results of Ray [30] and Schutz [32]. Another covariant Lie–Poisson bracket formalism is studied by Marsden, Montgomery, Morrison and Thompson [37].

Remark. Although there has been considerable progress in the development of noncanonical Hamiltonian structures in fluid dynamics since 1980, the idea of noncanonical Poisson brackets in continuum physics is not new. A catalog of noncanonical Poisson brackets of Lie–Poisson type used in continuum physics before 1980 should include: Landau [38] in the macroscopic theory of superfluids; Arnold [35] in incompressible flow; Dashen and Sharp [40] and Goldin and Sharp [41] in the classical theory of current algebras; Bialynicki-Birula and Iwinski [22] in special relativistic theories of both charged and neutral, but pressureless, fluids; and Iwinski and Turski [19] in special-relativistic, charged-fluid plasmas interacting both electromagnetically and thermodynamically. The last citation, of course, includes NRAF and SRAF.

In fact, noncanonical Poisson brackets were introduced into classical mechanics long ago by Lie [42] in his study of general composition laws satisfying the Jacobi identity. A modern perspective on Poisson structures is given in Weinstein

[43]. Applications of noncanonical Poisson brackets to the classical heavy top appear in Sudarshan and Mukunda [44] and Ratiu [45].

Contents. This paper is organized into four sections: section 2 treats NRAF; section 3 treats SRAF; section 4 treats GRAF; and section 5 presents conclusions and final comments. Each section is organized in a parallel fashion into four to six subsections that treat: (1) the starting equations of motion and notation; (2) introduction of a configuration space of position and velocity fields (these are the Lagrangian positions and velocities defined in *Eulerian* space); (3) an action principle in the Lagrangian configuration space; (4) the Legendre transformation to find the Hamiltonian density in the space of symplectic (canonically conjugate) fields; (5) definition of the map from the space of symplectic fields to the space of physical variables, such as the fluid mass density, specific entropy, and momentum density (this map determines the Lie–Poisson bracket and expresses the Hamiltonian density from step (4) as a functional in the space of physical variables); (6) calculation of the variational derivatives of the Hamiltonian with respect to the physical variables; (7) demonstration that using the Lie–Poisson bracket and Hamiltonian in the space of physical variables recovers the original equations of motion.

Form-invariance of the map in step (5) from the symplectic fields to the physical fields in Eulerian space in each of the three fluid theories results in form-invariance of the Lie–Poisson bracket obtained from the map. Consequently, the three fluid theories share the *same* Lie–Poisson bracket when expressed in the appropriate physical variables. Some of the implications of this shared structure for the dynamic stability of equilibrium solutions are discussed in section 5. Two appendices treat detailed calculations specific to the ADM formalism for GRAF. Except for making the factors of c and k explicit, we follow the notation and conventions of MTW.

2. Nonrelativistic adiabatic fluids

2.1. *Equations of motion and notation*

In adiabatic fluid dynamics, the fundamental variables are: the mass density ρ; the specific entropy η; and momentum density M_i, $i = 1, 2, \ldots, n$. The fluid moves through an n-dimensional Riemannian space, with positions x^i, $i = 1, 2, \ldots, n$, and metric tensor g_{ij}, $i, j = 1, 2, \ldots, n$.

In the nonrelativistic case, the fluid velocity v_i is related to the momentum density by

$$M_i = \sqrt{g}\,\rho v_i, \tag{2.1}$$

where $\sqrt{g} := \sqrt{\det g_{ij}}$. The Eulerian hydrodynamics equations are expressed as

$$\partial_t \rho = -\left(\rho v^i\right)_{;i} := -\frac{1}{\sqrt{g}}\left(\rho\sqrt{g}\,v^i\right)_{,i}, \tag{2.2a}$$

$$\partial_t \eta = -v^i \eta_{,i}, \tag{2.2b}$$

$$\partial_t v_i = -v^j v_{i;j} - \frac{1}{\rho} p_{,i} - \phi_{,i}, \tag{2.2c}$$

$$\Delta\phi := \left(\phi_{,j}\right)^{;j} = 4\pi G\rho, \tag{2.2d}$$

where partial time derivative is denoted by $\partial/\partial t$, partial space derivative is denoted by subscript comma (e.g., $\partial\eta/\partial x^i = \eta_{,i}$), covariant derivative compatible with the time-independent Riemannian metric g_{ij} (i.e., $\partial_t g_{ij} = 0$) is denoted by subscript semicolon $(\,;\,)$, and we sum on repeated indices over their indicated ranges. Indices are raised as for v^i in (2.2a) by the inverse metric tensor g^{ij}, which satisfies $g^{ij}g_{jk} = \delta^i_k$ and gives $v^i = g^{ij}v_j$. Eq. (2.2a) is the continuity equation expressing conservation of mass, and eq. (2.2b) is the adiabatic condition, so that each fluid element exchanges no heat with its surroundings. Eq. (2.2c) is the hydrodynamic motion equation expressed in covariant form. The fluid pressure p is determined as a function of ρ and η from a prescribed relation for the specific internal energy $e(\rho, \eta)$ (i.e. from an

equation of state) via the first law of thermodynamics,

$$de = e_\rho \, d\rho + e_\eta \, d\eta = \rho^{-2} p \, d\rho + T \, d\eta, \qquad (2.3)$$

where T is the temperature. The motion equation (2.2c) also includes a Newtonian gravitational potential, ϕ, determined by (2.2d) as a functional of ρ and a function of x, provided we choose boundary conditions such that the Laplacian operator Δ in (2.2d) is invertible.

2.2. *Action principle and Legendre transformation to canonical variables*

We show next that eqs. (2.2a–d) follows from a stationary variational principle, $\delta S = 0$, expressed in a space of variables different from those in (2.2a–d). Let us define Lagrange coordinates $q^A(x, t)$, $A = 1, 2, \ldots, n$, as time-dependent maps of spatial points x^i to reference points q^A. Then, density, specific entropy, and velocity are defined in terms of these Lagrange coordinates by the following expressions:

$$\rho(x, t)\sqrt{g(x)} = \bar{\rho}(\{q^A\}) \det(\partial q^A / \partial x^i)$$

$$=: \bar{\rho}(q) \det(q_i^A), \qquad (2.4a)$$

$$\eta(x, t) = \bar{\eta}(\{q^A\}) =: \bar{\eta}(q), \qquad (2.4b)$$

where $\bar{\rho}$ and $\bar{\eta}$ are prescribed functions of the argument $q := \{q^A\}$ determined by the values of the Lagrange coordinates q^A at some initial time, and q^A satisfies

$$\partial_t q^A + q_j^A v^j = 0, \qquad (2.4c)$$

with $q_j^A := \partial q^A / \partial x^j$. Regarding (q_j^A) as an invertible matrix allows eq. (2.4c) to be solved for v^j as

$$v^j = -(q^{-1})_A^j \dot{q}^A, \qquad (2.4d)$$

where $\dot{q}^A := \partial_t q^A$. We introduce definitions (2.4a–d) into the following action density:

$$\mathscr{L} = \sqrt{g} \left[\tfrac{1}{2} \rho |v|^2 - \rho e(\rho, \eta) - \tfrac{1}{2} \rho \phi \right]. \qquad (2.5)$$

Variation of the corresponding action $S = \int dt \, d^n x \, \mathscr{L}$ with respect to $\dot{q}^A$ determines the canonical momentum variable P_A conjugate to q^A as

$$P_A := \frac{\delta L}{\delta \dot{q}^A} = -\sqrt{g} \, \rho v_j (q^{-1})_A^j. \qquad (2.6)$$

Therefore, by taking the matrix product of (2.6) with q_j^A, the physical momentum density (2.1) is expressible as

$$M_j = \sqrt{g} \, \rho v_j = -P_A \partial_j q^A. \qquad (2.7)$$

We shall see that this type of momentum density relation is a generic feature of our approach.

One passes to the Hamiltonian formulation by Legendre transforming the action density (2.5), in which the velocities $\dot{q}^A$ need to be expressed in terms of (P_A, q^A). For this, we first rewrite (2.6) using (2.4d) as

$$P_A = (-\sqrt{g} \, \rho) \left[-g_{jk} (q^{-1})_B^k \dot{q}^B (q^{-1})_A^j \right]$$

$$= \sqrt{g} \, \rho \left[(q^{-1})_A^j g_{jk} (q^{-1})_B^k \right] \dot{q}^B$$

$$=: \sqrt{g} \, \rho (\Delta^{-1})_{AB} \dot{q}^B, \qquad (2.8)$$

where, in the notation of Künzle and Nester [8] (abbreviated as KN), the quantity

$$(\Delta^{-1})_{AB} = (q^{-1})_A^j g_{jk} (q^{-1})_B^k \qquad (2.9)$$

is the pull-back metric under the map q_A. Hence,

$$\sqrt{g} \, \rho \dot{q}^A = \Delta^{AB} P_B, \qquad (2.10)$$

where $\Delta^{AB} (\Delta^{-1})_{BC} = \delta_C^A$ and $g^{ab} g_{bc} = \delta_c^a$.

We have, therefore, the following relations:

$$\frac{1}{\rho\sqrt{g}}|\boldsymbol{P}|^2 := \frac{1}{\rho\sqrt{g}}P_A \Delta^{AB}P_B \text{ [by (2.10)]}$$

$$= P_A \dot{q}^A \text{ [by (2.4c)]}$$

$$= -P_A q_j^A v^j \text{ [by (2.7)]}$$

$$= v^j M_j \text{ [by (2.1)]}$$

$$= \rho\sqrt{g}|\boldsymbol{v}|^2 \text{ [by (2.1)]}$$

$$= \frac{1}{\rho\sqrt{g}}|\boldsymbol{M}|^2. \tag{2.11}$$

Equating the first and last of these expressions implies $|\boldsymbol{P}|^2 = |\boldsymbol{M}|^2$; so the momentum magnitudes are equal. Thus, the Hamiltonian that arises from Legendre transforming (2.5) and using (2.11) is

$$H = \int \mathrm{d}^n x \, \mathcal{H}$$

$$= \int \mathrm{d}^n x \left\{ \frac{|\boldsymbol{P}|^2}{2\rho\sqrt{g}} + \sqrt{g}\left[\rho e(\rho,\eta) + \tfrac{1}{2}\rho\phi\right]\right\}. \tag{2.12}$$

Note that this Hamiltonian is invariant under replacing P_A by $-P_A$, which we do now for later convenience. This replacement removes the minus signs in (2.6) and (2.7).

2.3. Map from canonical to physical fluid variables

At this stage in the standard Hamiltonian formalism for fluids in terms of canonically conjugate variables, the starting system should be shown to be expressible as $\partial_t F(\dot{p}, q) = \{H, F\}_c$ with canonical (symplectic) Poisson bracket

$$\{H, F\}_c = \int \mathrm{d}^n x \begin{pmatrix} \delta F/\delta P_A \\ \delta F/\delta q^A \end{pmatrix}^{\mathrm{t}}$$

$$\times \begin{pmatrix} 0 & -\delta_A^B \\ \delta_B^A & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta P_B \\ \delta H/\delta q^B \end{pmatrix}. \tag{2.13}$$

In other words, the canonical equations

$$\partial_t q^A = \frac{\delta H}{\delta P_A}, \quad \partial_t P_A = -\frac{\delta H}{\delta q^A}, \tag{2.14}$$

together with H given in terms of (P_A, q^A) by (2.12) should be shown to lead to the starting equations (after cumbersome algebraic manipulations, because of the complicated q^A dependence of H).

There is, however, an alternative and less cumbersome route for checking that the canonical equations (2.14) imply the physical motion equations (2.2a–d) via the relations (2.4a–d). This route, which we now explain, uses the noncanonical Hamiltonian formalism in the space of physical variables, and leads to considerable insight into the mathematical structure of the physical equations.

We seek a Hamiltonian description of the fluid motion equations [in the present case (2.2a–d)] directly in terms of the physical variables [in this case $\boldsymbol{M}, \rho, \eta$]. Such a description will result provided the following three requirements are satisfied:

1) The canonical Poisson bracket (2.13) induces a noncanonical Poisson bracket $\{\,,\}$ in the physical space, in this case by the map

$$M_i = P_A \partial_i q^A, \quad \hat{\rho} := \rho\sqrt{g} = \bar{\rho}(q)\det\left(q_i^A\right),$$

$$\eta = \bar{\eta}(q). \tag{2.15}$$

2) The Hamiltonian function in the canonical space can be expressed in terms of physical variables only, in the present case through (2.15) using (2.11).

3) This Hamiltonian function, H, expressed in terms of physical variables, generates in the physical space correct equations of motion according to the rule $F_t = \{H, F\}$, using the noncanonical Poisson bracket $\{\,,\}$ of requirement 1).

The second and third of these requirements are self explanatory. The first requirement amounts to checking the following formula (see, e.g., Holm

and Kupershmidt [23]):

$$\hat{\phi}(\boldsymbol{B}) = \frac{DZ}{DY} \boldsymbol{b} \left(\frac{DZ}{DY} \right)^{\dagger}, \qquad (2.16)$$

where: $\hat{\phi}$ is the map from the canonical space with coordinates Y, into the physical space with coordinates Z; $\boldsymbol{b}$ is the canonical matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\boldsymbol{B}$ is the Hamiltonian matrix in the physical space; $\hat{\phi}(\boldsymbol{B})$ is computed by applying the map $\hat{\phi}$ to each matrix element of $\boldsymbol{B}$; DZ/DY is the Fréchet derivative of the variables Z with respect to the variables Y; and the symbol $\dagger$ stands for the adjoint with respect to the measure $d^n x = dx^1 \wedge \cdots \wedge dx^n$.

In the present case, the Hamiltonian matrix in the space of physical variables which results from this procedure is (cf. Holm and Kupershmidt [23])

$$-\boldsymbol{B} = \begin{pmatrix} M_j \partial_i + \partial_j M_i & \hat{\rho} \partial_i & -\eta_{,i} \\ \partial_j \hat{\rho} & 0 & 0 \\ \eta_{,j} & 0 & 0 \end{pmatrix}, \qquad (2.17)$$

where $\eta_{,j} := (\partial \eta / \partial x^j)$ and ∂_j is regarded now as a differential operator. The Poisson bracket corresponding to the Hamiltonian matrix (2.17) is given by

$$\{H, F\} = -\int d^n x \left\{ \delta F / \delta M_i \left[(M_j \partial_i + \partial_j M_i) \delta H \right. \right.$$
$$\left. / \delta M_j + \hat{\rho} \partial_i \delta H / \delta \hat{\rho} - \eta_{,i} \delta H / \delta \eta \right]$$
$$+ \left[(\delta F / \delta \hat{\rho}) \partial_j \hat{\rho} \right.$$
$$\left. + (\delta F / \delta \eta) \eta_{,j} \right] \delta H / \delta M_j \right\}, \qquad (2.18)$$

where ∂_i operates on terms to the right of it.

To see how (2.17) arises, notice that applying formula (2.16) to the case when the matrix $\boldsymbol{b}$ is canonical as in (2.13), we obtain, for the $Z_i - Z_j$ entry of the Hamiltonian matrix $\hat{\phi}(\boldsymbol{B})$

$$\hat{\phi}(\boldsymbol{B})_{Z_i - Z_j} = \sum_A \left[\frac{DZ_i}{Dq^A} \left(\frac{DZ_j}{DP_A} \right)^{\dagger} - \frac{DZ_i}{DP_A} \left(\frac{DZ_j}{Dq^A} \right)^{\dagger} \right]. \qquad (2.19)$$

In the present case, the variables $\{Z\}$ are given in (2.15):

$$M_i = P_A q^A_{,i}, \quad \hat{\rho} = \bar{\rho}(q) \chi, \quad \eta = \bar{\eta}(q), \qquad (2.20)$$

where $q = \{q^A, A = 1, 2, \ldots, n\}$,

$$\chi = \det(q^A_{,i}), \qquad (2.21)$$

and $\bar{\rho}(q)$ and $\bar{\eta}(q)$ are given functions. Then, we have

$$\frac{DM_i}{Dq^A} = P_A \partial_i, \quad \frac{DM_i}{DP_A} = q^A_{,i},$$

$$\frac{D\hat{\rho}}{Dq^A} = \frac{\partial \bar{\rho}}{\partial q^A} \chi + \bar{\rho} \frac{\partial \chi}{\partial q^A_{,j}} \partial_j, \qquad (2.22)$$

$$\frac{D\hat{\rho}}{DP_A} = \frac{D\eta}{DP_A} = 0, \quad \frac{D\eta}{Dq^A} = \frac{\partial \bar{\eta}}{\partial q^A},$$

so that substituting eqs. (2.22) into (2.19) and using the standard identity

$$\chi^{-1} \sum_A \frac{\partial \chi}{\partial q^A_{,i}} q^A_{,j} = \delta^i_j, \qquad (2.23)$$

we obtain (2.17).

Remark. The Poisson bracket (2.18) is the natural Poisson bracket on the dual to the semidirect product Lie algebra $D \circledS [\Lambda^0 \oplus \Lambda^n]$ (see, e.g., Holm and Kupershmidt [23]), where $D = D(\mathbf{R}^n)$ represents vector fields on $\mathbf{R}^n$ (X_j denotes elements of D) and $\Lambda^k = \Lambda^k(\mathbf{R}^n)$ denotes k-forms on $\mathbf{R}^n$. D acts on itself by commutation of vector fields denoted by $[,]$ and acts upon Λ^k by Lie derivation, denoted, e.g., $X(\xi)$ for $\xi \in \Lambda^k$. The symbol $\circledS$ denotes semidirect product. The Lie algebraic commutator corresponding to the Poisson bracket (2.18) is, thus,

$$\left[(X; \xi^{(0)} \oplus \xi^{(n)}), (\bar{X}; \bar{\xi}^{(0)} \oplus \bar{\xi}^{(n)}) \right]$$
$$= \left([X, \bar{X}]; (X(\bar{\xi}^{(0)}) - \bar{X}(\xi^{(0)})) \oplus (X(\bar{\xi}^{(n)}) \right.$$
$$\left. - \bar{X}(\xi^{(n)})) \right). \qquad (2.18')$$

Dual coordinates are: M_i dual to $X_i \in D$, $\hat{\rho}$ dual to Λ^0, and η dual to Λ^n.

Poisson brackets such as (2.18) associated to the dual of a Lie algebra are called "Lie–Poisson brackets," see Marsden et al. [16] for an exposition and Weinstein [43] for more detail.

2.4. Hamiltonian formulation in physical fluid variables

Using relations (2.11), the Hamiltonian (2.12) can be expressed in terms of physical variables (2.15) as

$$H = \int d^n x \left[\frac{1}{2\hat{\rho}} |M|^2 + \hat{\rho} e(\hat{\rho}/\sqrt{g}, \eta) + \frac{1}{2}\hat{\rho}\phi \right].$$
(2.24)

The variational derivatives of this Hamiltonian are

$$\frac{\delta H}{\delta M_i} = v^i,$$

$$\frac{\delta H}{\delta \hat{\rho}} = -\frac{1}{2}|v|^2 + e + p/\rho + \phi,$$
(2.25)

$$\frac{\delta H}{\delta \eta} = \hat{\rho}T,$$

where T is temperature, defined in (2.3). Substituting the variational derivatives (2.25) into the Lie–Poisson bracket (2.18) leads immediately to

$$\partial_t \hat{\rho} = \{H, \hat{\rho}\} = -(\hat{\rho}v^j)_{,j},$$
(2.26a)

$$\partial_t \eta = \{H, \eta\} = -\eta_{,j}v^j,$$
(2.26b)

$$\partial_t M_i = \{H, M_i\} = -M_j v^j_{,i} - (M_i v^j)_{,j}$$
$$-\hat{\rho}\left(-\frac{1}{2}|v|^2 + e + p/\rho + \phi\right)_{,i} + \hat{\rho}T\eta_{,i}.$$
(2.26c)

Eqs. (2.26a, b) reproduce (2.2a, b), the continuity equation and adiabatic condition, respectively. Eq. (2.26c) can be re-expressed in terms of velocity to recover (2.2c). Using (2.26a) and (2.3) we find from (2.26c) that

$$\partial_t v_i = -\frac{1}{\rho}p_{,i} - \phi_{,i} - v^j v_{i,j} + \frac{1}{2}g_{jk,i}v^j v^k.$$
(2.27)

We rearrange the last two terms in (2.27) as follows. Upon setting

$$y_i := v^j v_{i,j} - \frac{1}{2}g_{jk,i}v^j v^k = v^j(g_{ik}v^k)_{,j} - \frac{1}{2}g_{jk,i}v^j v^k$$
$$= g_{ik}v^j v^k_{,j} + g_{ik,j}v^j v^k - \frac{1}{2}g_{jk,i}v^j v^k,$$
(2.28)

symmetrizing the middle term in (2.28) gives

$$y_i = g_{ik}v^j v^k_{,j} + \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i})v^j v^k,$$
(2.29)

which we recognize upon raising indices as

$$y^i = g^{im}y_m = v^j v^i_{,j} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} v^j v^k =: v^j v^i_{;j},$$
(2.30)

i.e., the covariant derivative of v^i in the direction v^j. Hence, the motion equation (2.26c) becomes

$$\partial_t v^i = -\frac{1}{\rho}g^{ij}p_{,j} - g^{ij}\phi_{,j} - v^j v^i_{;j},$$
(2.31)

which reproduces (2.2c).

Thus, the Lie–Poisson bracket (2.18) and Hamiltonian (2.24) yield the nonrelativistic adiabatic fluid (NRAF) eqs. (2.2a–c) directly in terms of the Eulerian physical variables. Since we have already seen via (2.16) that the map (2.15) is canonical (in the sense of preserving Poisson brackets), the Hamilton equations (2.14) will also imply the fluid equations (2.2a–c).

2.5. Remark on Casimir functionals of the Lie–Poisson bracket (2.18)

First, observe that using (2.26a) allows (2.26c) to be re-expressed as

$$\partial_t v_i = -(v^j v_{i,j} + v_j v^j_{,i})$$
$$-\left(-\frac{1}{2}|v|^2 + e + p/\rho + \phi\right)_{,i} + T\eta_{,i}.$$
(2.32)

Notice that the first term on the right-hand side of (2.32) is the Lie–derivative with respect to velocity of the circulation one-form $v_i dx^i$, namely

$$\mathcal{L}_v(v_i dx^i) = (v^j v_{i,j} + v_j v^j_{,i}) dx^i.$$
(2.33)

Consequently, the motion equation (2.32) can be written as

$$\partial_t(v_i\,dx^i) = -\mathscr{L}_v(v_i\,dx^i)$$
$$-d\left(-\tfrac{1}{2}|v|^2 + e + p/\rho + \phi\right) + T\,d\eta \qquad (2.34)$$

in terms of the Lie derivative, $\mathscr{L}_v$, and the spatial exterior derivative, d. Likewise, eqs. (2.26a,b) can be expressed as

$$\partial_t(\hat{\rho}\,d^n x) = -\mathscr{L}_v(\hat{\rho}\,d^n x), \qquad (2.35a)$$

$$\partial_t\eta = -\mathscr{L}_v\eta. \qquad (2.35b)$$

Now, taking the exterior product of d times (2.34) with d times (2.35b) (using the properties $d^2 = 0 = [d, \mathscr{L}_v]$) implies that the three-form $d(v_i\,dx^i) \wedge d\eta =: \hat{\rho}\Omega\,d^3x$, with $\Omega := \hat{\rho}^{-1}\,\text{curl}\,v\cdot\nabla\eta$, is conserved along flow lines, i.e.,

$$\partial_t(\hat{\rho}\Omega\,d^3x) = -\mathscr{L}_v(\hat{\rho}\Omega\,d^3x). \qquad (2.36)$$

Consequently, in three dimensions ($n = 3$) we may use (2.35a) and (2.36) to show that

$$\partial_t\Omega = -\mathscr{L}_v\Omega, \qquad (2.37)$$

for the scalar function Ω defined above to be

$$\Omega = \hat{\rho}^{-1}\,\text{curl}\,v\cdot\nabla\eta. \qquad (2.38)$$

Thus, we have the following conservation law for adiabatic fluid flow in a three-dimensional Riemannian space:

$$\partial_t C = 0, \quad C = \int d^3x\,\hat{\rho}\Phi(\eta,\Omega), \qquad (2.39)$$

for an arbitrary function Φ of the two indicated arguments, η and Ω.

The presence of the arbitrary function Φ in (2.39) is a clue that this conservation law is kinematical; depending only on having expressed the Hamiltonian in the space of physical variables, rather than depending on the dynamics generated by the particular choice of Hamiltonian (2.24). Indeed, the conserved functional C in (2.39) is a "Casimir" in the sense that

$$\{C, F\} = 0, \quad \forall F(\{\hat{\rho}, \eta, M_i\}). \qquad (2.40)$$

That is, C is in the kernel of the Lie–Poisson bracket (2.18) and, thus, is conserved *independently of the choice of Hamiltonian* in the space of physical fluid variables $\{\eta, \hat{\rho}, M_i\}$. The advected quantity Ω is the generalization for three-dimensional Riemannian space and compressible fluids of Ertel's invariant, the so-called potential vorticity (see, e.g., Kochin, Kibel, and Roze [46]).

The conservation law (2.39) can also be understood as resulting via Noether's theorem from the symmetry of the action density (2.5) under the transformations in the Lagrangian configuration space $(q^A, \dot{q}^A)$ that leave invariant the Eulerian variables $\{\hat{\rho}, \eta, M_i\}$. Such so-called "trivial" transformations (in the nomenclature of Friedman and Schutz [47]) can be considered as gauge transformations under the group of diffeomorphisms of the Lagrangian fields q^A preserving the value of the density $\hat{\rho}$, specific entropy η, and velocity v^i at each Eulerian point (cf. Marsden, Ratiu, and Weinstein [14]). The corresponding conservation law is then (2.39). Explicitly, the allowed infinitesimal transformations are those satisfying (see Holm [48])

$$\delta\rho = \frac{\rho}{\bar{\rho}(q)}\frac{\partial}{\partial q^A}[\bar{\rho}(q)\delta q^A] = 0, \qquad (2.41a)$$

$$\delta\eta = \frac{\partial\bar{\eta}}{\partial q^A}\delta q^A = 0, \qquad (2.41b)$$

$$\delta v^i = -(q^{-1})^i_A\frac{\partial\delta q^A}{\partial t} = 0, \qquad (2.41c)$$

That is, the action

$$S = \int dt\,d^n x\,\mathscr{L}, \qquad (2.42)$$

with $\mathscr{L}$ given in (2.5) is invariant under the infinitesimal transformation

$$q^A(x, t) \to \bar{q}^A(x, t) = q^A(x, t) + \delta q^A(x, t),$$
$$(2.43)$$

with

$$\delta q^A = \frac{1}{\bar{\rho}(q)}\mathrm{curl}_q\left[f(\bar{\eta})\nabla_q a(q)\right] \qquad (2.44)$$

satisfying (2.41a–c), where $f(\bar{\eta})$ and $a(q)$ are arbitrary functions, and subscript q on ∇_q denotes a gradient in Lagrangian coordinates. The corresponding conserved density is then $\bar{\rho}\Phi(\Omega)$ with Ω given in (2.38) and Φ an arbitrary function.

Recently, Casimirs such as (2.39) have been used to study the Lyapunov stability of fluid and plasma equilibria in a variety of situations, see, e.g., Abarbanel et al. [49, 50], Arnold [51, 52], Hazeltine et al. [53], Holm, Marsden, Ratiu and Weinstein [13, 54], and Holm and Kupershmidt [55]. Further comments concerning Casimirs and Lyapunov stability are given at the conclusion of this paper, in section 5.

3. Special relativistic adiabatic fluids

3.1. *Equations of motion and notation*

The special relativistic generalization of the adiabatic fluid equations (2.2a–c) in Riemannian space is, in Lorentz-covariant form,

$$\left(\rho_f u^\mu\right)_{;\mu} = 0, \qquad (3.1a)$$

$$u^\mu \eta_{f,\mu} = 0, \qquad (3.1b)$$

$$T^{\mu\nu}_{\ \ ;\nu} = 0, \qquad (3.1c)$$

where u^μ, with $\mu = 0, 1, \ldots, n$ (Greek indices run from 0 to n), denotes the timelike Lorentz vector for the fluid velocity, which becomes $u^0 = 1$, $u^i = 0$, $i = 1, 2, \ldots, n$, (Latin indices run from 1 to n) in the reference frame of the fluid. The vector u^μ satisfies

$$g_{\mu\nu}u^\mu u^\nu = -1. \qquad (3.2)$$

The space–time metric tensor $g_{\mu\nu}$ is given by the expression $-\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ for the proper time interval, $x^0 = ct$ being the real timelike coordi-

nate. In (3.1), covariant derivatives with respect to the space–time metric $g_{\mu\nu}$ are denoted by a semi-colon subscript (;). Ordinary partial derivatives are denoted by a comma subscript (,). Subscript f denotes variables as measured in the reference frame moving with the fluid. For example, ρ_f is proper mass density and η_f is proper specific entropy, each in the fluid frame. The quantity $T^{\mu\nu}$ in eq. (3.1) is the energy-momentum tensor, given by

$$T^{\mu\nu} = \rho_f c^2 w u^\mu u^\nu + p_f g^{\mu\nu}, \qquad (3.3)$$

where c is the speed of light, p_f is the pressure in the fluid frame, and w is the relativistic specific enthalpy, defined as

$$w = 1 + \left(e_f + p_f/\rho_f\right)c^{-2}, \qquad (3.4)$$

with specific internal energy in the fluid frame e_f prescribed by an equation of state $e_f(\rho_f, \eta_f)$ satisfying

$$\mathrm{d}e_f = T_f\,\mathrm{d}\eta_f + \left(\rho_f\right)^{-2}p_f\,\mathrm{d}\rho_f. \qquad (3.4')$$

Eq. (3.1a) expresses relativistic mass conservation, eq. (3.1b) is the adiabatic condition for the relativistic fluid, and (3.1c) expresses the covariant conservation laws for energy and momentum. There is a well-known redundancy among eqs. (3.1a–c): a linear combination of eqs. (3.1a, b) can be obtained from the projection of (3.1c) along u_μ (see, e.g., Taub [56]). Contracting u_μ with (3.1c) using $T^{\mu\nu}$ given by (3.3) and the condition $u_\mu u^\mu_{\ ;\nu} = 0$ (implied by (3.2)) contributes the relation

$$
\begin{aligned}
0 &= u_\mu T^{\mu\nu}_{\ \ ;\nu} \\
&= u_\mu c^2 w u^\mu\left(\rho_f u^\nu\right)_{;\nu} + \rho_f u^\nu\left(c^2 w u^\mu\right)_{;\nu} u_\mu + u^\nu p_{f,\nu} \\
&= -c^2 w\left(\rho_f u^\nu\right)_{;\nu} - \rho_f u^\nu\left(c^2 w\right)_{,\nu} + u^\nu p_{f,\nu} \\
&= -c^2 w\left(\rho_f u^\nu\right)_{;\nu} - \rho_f T_f\left(u^\nu \eta_{f,\nu}\right), \qquad (3.5)
\end{aligned}
$$

where we have used the definition of w in (3.4) and the thermodynamic relation (3.4') in the fluid frame for the last step. According to (3.5), the projection of (3.1c) parallel to u_μ and one of either

(3.1a), or (3.1b), implies the remaining equation in (3.1a, b).

Expanding (3.1c) using the chain rule gives

$$
\begin{aligned}
0 &= T^{\mu\nu}{}_{;\nu} \\
&= c^2 w u^\mu (\rho_f u^\nu)_{;\nu} + \rho_f u^\nu (c^2 w u^\mu)_{;\nu} + g^{\mu\nu} p_{f,\nu}.
\end{aligned}
\tag{3.6}
$$

Upon lowering the free index μ in (3.6) and using (3.1a), we have (3.1c) in the form we shall use:

$$
0 = T^\nu_{\mu;\nu} = \rho_f u^\nu \left(c^2 w u_\mu \right)_{;\nu} + p_{f,\mu}.
\tag{3.7}
$$

The n spatial components of (3.7) comprise the relativistic counterpart. of the motion equation (2.2c) (excluding the Newtonian potential); the time component of (3.7) is a consequence of the space components.

For the purposes of Hamiltonian formulation, we express eqs. (3.1a, b) and the n space components of eq. (3.7) as a dynamical system in a fixed frame. This we choose to be the laboratory frame, in which u^μ becomes $u^0 = \gamma$, $u^i = \gamma v^i/c$, with

$$
\gamma := \left(1 - v_i v^i/c^2\right)^{-1/2},
\tag{3.8}
$$

where $v_i = g_{ij} v^j$ and g_{ij} denotes the (fixed, time-independent) spatial Riemannian metric. In the laboratory frame, fluid state variables will be unadorned and related to their fluid-frame counterparts by $\eta = \eta_f$ and $\rho = \gamma \rho_f$. Upon using $u^\mu = \gamma(1, v^i/c)$, $\partial_t g_{ij} = 0$, and the identity $(\rho_f u^\mu)_{;\mu} = (\sqrt{g}\, \rho_f u^\mu)_{,\mu}/\sqrt{g}$, the dynamical system resulting from (3.1) in the laboratory frame becomes, with $\hat\rho = \gamma \rho_f \sqrt{g}$, $\eta = \eta_f$, $\sqrt{g} = \sqrt{\det(g_{ij})}$,

$$
\partial_t \hat\rho = -\left(\hat\rho v^i\right)_{,i},
\tag{3.9a}
$$

$$
\partial_t \eta = -\eta_{,i} v^i,
\tag{3.9b}
$$

$$
\partial_t (\gamma w v_i) = -v^j (\gamma w v_i)_{;j} - \frac{\sqrt{g}}{\hat\rho} p_{f,i},
\tag{3.9c}
$$

where subscript semicolon (;) adjoined to Latin indices denotes covariant derivative compatible with the Riemannian metric tensor g_{ij}, as in section 2. In the nonrelativistic limit ($c^{-1} \to 0$), $\gamma = 1 + \mathcal{O}(c^{-2})$, $w = 1 + \mathcal{O}(c^{-2})$, and as c^{-2} tends to zero each equation in (3.9a–c) tends to its nonrelativistic counterpart in (2.2a–c).

3.2. *Action principle and Legendre transformation to canonical variables*

As an auxiliary step in constructing the Hamiltonian formalism for eqs. (3.9a–c), we introduce a stationary variational principle, $\delta S = 0$, expressed in a space of Lagrangian fields $q^A(x, t)$, $A = 1, 2, \ldots, n$, just as in section 2. First, we define the variables $\hat\rho$, η, v^i in terms of these Lagrangian fields, via expressions analogous to (2.4a–d), namely,

$$
\hat\rho(x, t) = \bar\rho(q) \det(q_i^A), \quad \eta(x, t) = \bar\eta(q), \quad \dot q^A + q_j^A v^j = 0, \quad v^j = -\left(q^{-1}\right)^j_A \dot q^A,
\tag{3.10a,b,c,d}
$$

where $q_i^A = \partial q^A/\partial x^i$, $\dot q^A = \partial q^A/\partial t$, $(q^{-1})^j_A$ is the matrix inverse of q_i^A [so that $(q^{-1})^j_A q_i^A = \delta_i^j$ and $(q^{-1})^j_A q_j^B = \delta_A^B$], and $\bar\rho(q)$ and $\bar\eta(q)$ are prescribed functions of $q := \{q^A\}$. We introduce definitions (3.10a–d) into the following Lagrangian density:

$$
\mathcal{L} = -\sqrt{g}\, \varepsilon(\rho_f, \eta_f).
\tag{3.11}
$$

Here, ε is the total energy density for special relativity,

$$
\varepsilon = \rho_f \left(c^2 + e(\rho_f, \eta_f)\right),
\tag{3.12}
$$

so that, by (3.4′),

$$d\varepsilon = \left(c^2 + e_f + p_f/\rho_f\right)d\rho_f + \rho_f T_f \, d\eta_f = c^2 w \, d\rho_f + \rho_f T_f \, d\eta_f. \tag{3.13}$$

Variation of the action $S = \int dt \, d^n x \, \mathscr{L}$ from (3.11) with respect to $\dot{q}^A$ yields the canonical momentum variable P_A, conjugate to q^A, as

$$P_A := \frac{\delta L}{\delta \dot{q}^A} = -\sqrt{g}\, \left.\frac{\partial \varepsilon}{\partial \rho_f}\right|_{\eta_f} \left.\frac{\partial \rho_f}{\partial \gamma^{-1}}\right|_{\hat{\rho}} \frac{\partial \gamma^{-1}}{\partial v^j} \frac{\partial v^j}{\partial \dot{q}^A} = -\sqrt{g}\, c^2 w \left(\hat{\rho}/\sqrt{g}\right)\left(-\frac{\gamma}{c^2} v_j\right)\left(-(q^{-1})^j_A\right)$$

$$= -\hat{\rho}\gamma w v_j (q^{-1})^j_A =: -\hat{\theta} v_j (q^{-1})^j_A, \tag{3.14}$$

where we have used (3.13), the definitions $\rho_f = \hat{\rho}(\sqrt{g}\,\gamma)^{-1}$ and $\gamma^{-1} = (1 - v_i v^i/c^2)^{1/2}$, and have introduced the notation

$$\hat{\theta} := \hat{\rho}\gamma w = \sqrt{g}\, \rho_f \gamma^2 w. \tag{3.15}$$

By taking the matrix product of eq. (3.14) with q^A_i, we find the physical momentum density in the laboratory frame, M_i, to be (cf. (2.7))

$$M_i := \frac{\sqrt{g}}{c} T_i^0 \text{ [by (3.3)]} = \hat{\theta} v_i \text{ [by (3.14)]} = -P_A q^A_i. \tag{3.16}$$

We now seek the canonical Hamiltonian formalism in terms of (P_A, q^A). To solve for $\dot{q}^A$ in terms of (P_A, q^A), we first rewrite P_A in (3.14) using (3.10d) as

$$P_A = [-\hat{\theta}]\left[-g_{jk}(q^{-1})^k_B \dot{q}^B\right]\left[(q^{-1})^j_A\right] = \hat{\theta}(\Delta^{-1})_{AB} \dot{q}^B, \tag{3.17}$$

where Δ^{-1} is defined as in (2.9). Hence, with Δ^{AB} defined by $\Delta^{AB}(\Delta^{-1})_{BC} = \delta^A_C$, we have

$$\Delta^{AB} P_B = \hat{\theta} \dot{q}^A \tag{3.18}$$

and we find the following relations:

$$\hat{\theta}^{-1}|P|^2 := \hat{\theta}^{-1} P_A \Delta^{AB} P_B \text{ [by (3.18)]} = P_A \dot{q}^A \text{ [by (3.10c)]} = -P_A q^A_i v^i \text{ [by (3.16)]}$$

$$= M_i v^i \text{ [by (3.16)]} = \hat{\theta}|v|^2 \text{ [by (3.16)]} = \hat{\theta}^{-1}|M|^2, \tag{3.19}$$

where we have introduced the momentum magnitudes $|P|^2 = P_A \Delta^{AB} P_B$ and $|M|^2 = M_i g^{ij} M_j$ related by $|P|^2 = |M|^2$, as shown in (3.19) [cf. (2.11)].

Using (3.18), we see that $P_A \dot{q}^A = \hat{\theta}^{-1}|P|^2$, so that the Hamiltonian obtained by Legendre transforming the action density (3.11) takes the form

$$H = \int d^n x \, \mathscr{H} = \int d^n x \left[\hat{\theta}^{-1}|P|^2 + \sqrt{g}\, \varepsilon(\rho_f, \eta_f)\right]. \tag{3.20}$$

Here $\hat{\theta}$ is given in (3.15) and depends on the relativistic factor, γ, and the other dynamical variables. By the

definition of γ in (3.8) we have

$$1 - \gamma^{-2} = c^{-2}v_i v^i \ \left[\text{by (3.10d) and (2.9)}\right] = c^{-2}(\Delta^{-1})_{AB}\dot{q}^A\dot{q}^B \ \left[\text{by (3.18)}\right]$$

$$= (c\hat{\theta})^{-2}|P|^2 = (c\hat{\theta})^{-2}|M|^2. \tag{3.21}$$

Thus, γ is expressible in terms of $\hat{\theta}$ and the momentum magnitudes. Consequently, either for the canonical variables (P_A, q^A), or for the noncanonical physical variables, $\{\hat{\rho}, \eta, M_i\}$, the only implicit dependence in the Hamiltonian (3.20) is in $\hat{\theta}$.

3.3. *Map from canonical to physical fluid variables*

We could now consider the relation to the fluid eqs. (3.9a–c) of the canonical equations that follow from the Hamiltonian (3.20) in the space of canonical variables (P_A, q^A). However, the additional complexity due to implicit dependence in $\hat{\theta}$ makes this task even less perspicuous than in the earlier, nonrelativistic case in section 2. Instead, just as in that earlier case, we shall seek a Hamiltonian description of the fluid motion equations [in the present case, (3.9a–c)] directly in terms of the physical variables [in the present case, the relativistic laboratory-frame quantities $\{\hat{\rho}, \eta, M_i\}$]. For this, we first replace P_A by $(-P_A)$ in the Hamiltonian (3.20) (which leaves the Hamiltonian invariant) and in eq. (3.16) (which conveniently changes the last minus sign in (3.16) to plus). Then, collecting equations (3.10a, b) and the revised eq. (2.16) results in the following analog of the Lagrange-to-Euler map:

$$M_i = P_A \partial_i q^A, \quad \hat{\rho} = \bar{\rho}(q) \det(q_i^A), \quad \eta = \bar{\eta}(q), \tag{3.22}$$

which is identical in form to the map (2.15), but now the earlier nonrelativistic variables on the left-hand sides of the map are replaced by relativistic, laboratory-frame variables. The Hamiltonian functional (3.20) in the canonical space can be expressed in terms of physical variables only, through the last equality in (3.21), resulting in

$$H = \int \mathrm{d}^n x \ \left[(|M|^2/\hat{\theta} + \sqrt{g}\,\varepsilon(\rho_f, \eta_f)\right], \tag{3.23}$$

where $\rho_f = \hat{\rho}(\gamma\sqrt{g})^{-1}$ and $\eta_f = \eta$. The Hamiltonian density in (3.23) is related to T^{00}, the time-time component of $T^{\mu\nu}$ in (3.3), by

$$\mathcal{H} = |M|^2/\hat{\theta} + \sqrt{g}\,\varepsilon(\rho_f, \eta_f) = \sqrt{g}\,T^{00}. \tag{3.24}$$

This relation is shown by a direct computation, which produces a convenient expression for (3.23) as a bonus. In the laboratory frame, we have $g^{00} = -1$, so that

$$\sqrt{g}\,T^{00} \ \left[\text{by (3.3)}\right] = \sqrt{g}\,(\rho_f c^2 w \gamma^2 - p_f) = \sqrt{g}\,\left[(\gamma^2 - 1)\rho_f c^2 w + \rho_f c^2 w - p_f\right] \ \left[\text{by (3.4) and (3.12)}\right]$$

$$= \sqrt{g}\,\left[(\gamma^2 - 1)\rho_f c^2 w + \varepsilon(\rho_f, \eta_f)\right] \ \left[\text{by (3.15) and (3.21)}\right]$$

$$= \sqrt{g}\,\left[\frac{|M|^2/c^2}{(\hat{\rho}w)^2}\rho_f c^2 w + \varepsilon(\rho_f, \eta_f)\right] \ \left[\text{by } \hat{\rho} = \gamma\sqrt{g}\,\rho_f\right]$$

$$= \sqrt{g}\,\left[\frac{|M|^2}{\sqrt{g}\,\hat{\rho}\gamma w} + \varepsilon(\rho_f, \eta_f)\right] \ \left[\text{by (3.15)}\right] = |M|^2/\hat{\theta} + \sqrt{g}\,\varepsilon(\rho_f, \eta_f).$$

This computation proves relation (3.24), demonstrates the physical significance of the Hamiltonian density, $\mathscr{H}$, and results upon using (3.15) in the following convenient expression for the Hamiltonian (3.23):

$$H = \int d^n x \sqrt{g} \, T^{00} = \int d^n x \left(c^2 \hat{\theta} - \sqrt{g} \, p_f \right). \tag{3.25}$$

3.4. Hamiltonian formulation in physical fluid variables

Just as in section 2, the "Lagrange-to-Euler" map (3.22) takes the canonical Hamiltonian matrix in (2.13) to that in (2.17), resulting in the Lie–Poisson bracket (2.18). The variational derivatives of the Hamiltonian (3.23) with respect to the physical variables are shown below to be

$$\frac{\delta H}{\delta M_i} = v^i, \tag{3.26a}$$

$$\frac{\delta H}{\delta \hat{\rho}} = c^2 w / \gamma, \tag{3.26b}$$

$$\frac{\delta H}{\delta \eta} = \sqrt{g} \, \rho_f T_f. \tag{3.26c}$$

We shall soon see that substituting the variational identities (3.26a–c) into $\partial_t F = \{ H, F \}$ with Lie–Poisson bracket (2.18) and Hamiltonian (3.23) will yield the relativistic adiabatic system in the laboratory frame (3.9a–c). However, first we show how these identities arise. By (3.4) and (3.4'), we find, as a preliminary step, that

$$c^2 dw = \rho_f^{-1} \frac{\partial p_f}{\partial \rho_f} d\rho_f + \left(\rho_f^{-1} \frac{\partial p_f}{\partial \eta_f} + T_f \right) d\eta_f. \tag{3.27}$$

Also, from (3.15) and (3.21) we have

$$1 - (\hat{\rho} w / \hat{\theta})^2 = 1 - \gamma^{-2} = |v|^2 / c^2 = |M|^2 / (c\hat{\theta})^2, \tag{3.28}$$

so that

$$\hat{\theta} = \sqrt{(\hat{\rho} w)^2 + |M|^2 / c^2}. \tag{3.29}$$

Using formulae (3.27), (3.29), and $\hat{\rho} = \rho_f \gamma \sqrt{g}$, $\eta =$

η_f we obtain from (3.25)

$$\frac{\delta H}{\delta M_i} = c^2 \frac{\partial \hat{\theta}}{\partial M_i} - \sqrt{g} \frac{\partial p_f}{\partial M_i}$$

$$= \frac{c^2}{\hat{\theta}} \left(M^i / c^2 + \hat{\rho}^2 w \frac{\partial w}{\partial \rho_f} \frac{\partial \rho_f}{\partial M_i} \right) - \sqrt{g} \frac{\partial p_f}{\partial \rho_f} \frac{\partial \rho_f}{\partial M_i}$$

$$= M^i / \hat{\theta} + \frac{\partial \rho_f}{\partial M_i} \left(\frac{c^2 \hat{\rho}}{\gamma} \frac{\partial w}{\partial \rho_f} - \sqrt{g} \frac{\partial p_f}{\partial \rho_f} \right)$$

$$= M^i / \hat{\theta} = v^i, \tag{3.30}$$

which proves (3.26a). Next, by (3.27) and $\hat{\rho} = \rho_f \gamma \sqrt{g}$,

$$\frac{\delta H}{\delta \hat{\rho}} = \frac{c^2}{\hat{\theta}} \hat{\rho} w \frac{\partial (\hat{\rho} w)}{\partial \hat{\rho}} - \sqrt{g} \frac{\partial p_f}{\partial \hat{\rho}}$$

$$= \frac{c^2}{\gamma} \left(w + \hat{\rho} \frac{\partial w}{\partial \hat{\rho}} \right) - \sqrt{g} \frac{\partial p_f}{\partial \hat{\rho}}$$

$$= c^2 w / \gamma + \sqrt{g} \left(c^2 \rho_f \frac{\partial w}{\partial \rho_f} - \frac{\partial p_f}{\partial \rho_f} \right) \frac{\partial \rho_f}{\partial \hat{\rho}}$$

$$= c^2 w / \gamma, \tag{3.31}$$

which is (3.26b). Finally

$$\frac{\delta H}{\delta \eta} = \frac{c^2}{\hat{\theta}} \hat{\rho} w \frac{\partial \hat{\rho} w}{\partial \eta} - \sqrt{g} \frac{\partial p_f}{\partial \eta}$$

$$= \frac{c^2}{\gamma} \hat{\rho} \frac{\partial w}{\partial \eta} - \sqrt{g} \frac{\partial p_f}{\partial \eta}$$

$$= \sqrt{g} \left(\rho_f c^2 \frac{\partial w}{\partial \eta} - \frac{\partial p_f}{\partial \eta} \right)$$

$$= \sqrt{g} \, \rho_f T_f, \tag{3.32}$$

which is (3.26c).

Substituting the variational derivatives (3.26a–c) into the Lie–Poisson bracket (2.18) leads im-

mediately to

$$\partial_t \hat{\rho} = \{ H, \hat{\rho} \} = -(\hat{\rho} v^j)_{,j}, \tag{3.33}$$

$$\partial_t \eta = \{ H, \eta \} = -\eta_{,j} v^j, \tag{3.34}$$

$$\begin{aligned}
\partial_t M_i &= \{ H, M_i \} \\
&= -M_j v^j_{,i} - (M_i v^j)_{,j} - \gamma^{-1} \hat{\rho} c^2 w_{,i} \\
&\quad - \hat{\rho} c^2 w (\gamma^{-1})_{,i} + \gamma^{-1} \hat{\rho} T_f \eta_{,i}.
\end{aligned} \tag{3.35a}$$

Substituting (3.27) into (3.35a) leads to

$$\begin{aligned}
\partial_t M_i &= -M_j v^j_{,i} - (M_i v^j)_{,j} - \sqrt{g}\, p_{f,i} \\
&\quad - \hat{\rho} c^2 w (\gamma^{-1})_{,i}.
\end{aligned} \tag{3.35b}$$

Using (3.33) and the definition $M_i = \hat{\rho} \gamma w v_i$ in eq. (3.35b) yields

$$\begin{aligned}
\partial_t (\gamma w v_i) &= -\gamma w v_j v^j_{,i} - v^j (\gamma w v_i)_{,j} \\
&\quad - \frac{\sqrt{g}}{\hat{\rho}} p_{f,i} - c^2 w (\gamma^{-1})_{,i}.
\end{aligned} \tag{3.36}$$

Now, $(\gamma^{-1})_{,i}$ in (3.36) is given by

$$(\gamma^{-1})_{,i} = -\frac{\gamma}{c^2} v_j v^j_{,i} - \frac{\gamma}{2c^2} v^j v^k g_{jk,i}, \tag{3.37}$$

so that (3.36) becomes

$$\begin{aligned}
\partial_t (\gamma w v_i) &= -v^j (\gamma w v_i)_{,j} - \frac{\sqrt{g}}{\hat{\rho}} p_{f,i} \\
&\quad + \tfrac{1}{2} \gamma w v^j v^k g_{jk,i}.
\end{aligned} \tag{3.38}$$

Then, rearranging terms in (3.38) as in (2.28–30) gives

$$\partial_t (\gamma w v_i) = -v^j (\gamma w v)_{;j} - \frac{\sqrt{g}}{\hat{\rho}} p_{f,i}, \tag{3.39}$$

which is the equation of motion (3.9c).

Thus, the Lie–Poisson bracket (2.18) and Hamiltonian (3.25) yield the special relativistic adiabatic fluid eqs. (3.9a–c), directly in terms of the Eulerian physical variables, $\{ \hat{\rho}, \eta, M_i \}$. This implies, because the map (3.22) is canonical (i.e.,

preserves Poisson brackets according to (2.16)), that the Lagrangian equations in the canonically conjugate variables (P_A, q^A) are also equivalent to eqs. (3.9a–c).

Remarks. A) The Lie–Poisson bracket (2.18) for SRAF appears in Iwinski and Turski [19], where electromagnetic interactions via Maxwell's equations are included, as well. (Iwinski and Turski [19] also presents a Lie–Poisson bracket for the SR Maxwell–Vlasov system, which is not discussed here.) The present derivation of this Lie–Poisson bracket for SRAF illustrates its relation to the corresponding symplectic bracket in Lagrangian fields, $q^A(x, t)$ and $P_A(x, t)$.

B) Using (2.27), the motion eq. (3.36) can be expressed in Lie-derivative form as

$$\begin{aligned}
\partial_t (\gamma w v_i \, dx^i) &= -\mathcal{L}_v (\gamma w v_i \, dx^i) - d(c^2 w/\gamma) \\
&\quad + \gamma^{-1} T_f \, d\eta,
\end{aligned} \tag{3.40}$$

where d denotes exterior derivative and $\mathcal{L}_v$ is the Lie derivative with respect to the vector field $v^i \partial_i$. Similarly, eqs. (3.9a, b) can be expressed as

$$\partial_t (\hat{\rho} \, d^n x) = -\mathcal{L}_v (\hat{\rho} \, d^n x), \tag{3.41a}$$

$$\partial_t \eta = -\mathcal{L}_v \eta. \tag{3.41b}$$

As a consequence of the properties $d^2 = 0 = [d, \mathcal{L}_v]$ of the exterior and Lie derivatives, and the antisymmetry of the exterior product, we find from (3.40) and (3.41a) that the three-form $d(\gamma w v_i \, dx^i) \wedge d\eta$ is preserved along flow lines, i.e.

$$\partial_t [d(\gamma w v_i \, dx^i) \wedge d\eta] = -\mathcal{L}_v [d(\gamma w v_i \, dx^i) \wedge d\eta]. \tag{3.42}$$

Consequently, in three dimensions ($n = 3$) we find as in section 2.5 that

$$\partial_t \Omega = -\mathcal{L}_v \Omega, \tag{3.43}$$

where the scalar Ω is now defined to be

$$\Omega = (\hat{\rho})^{-1} \operatorname{curl}(\gamma w v) \cdot \nabla \eta. \tag{3.44}$$

Thus, for $n = 3$ we have the following conservation law for special relativistic adiabatic fluids, $\partial C / \partial t = 0$ for

$$C = \int \mathrm{d}^3 x \, \hat{\rho} \Phi(\eta, \Omega), \qquad (3.45)$$

for an arbitrary function Φ of the two indicated variables in (3.45). The conserved functional C in (3.45) is the special relativistic version of the "Casimir" functional (2.39) for the Lie–Poisson bracket (2.18). Its use for determining stability criteria for special relativistic fluids is discussed in Holm and Kupershmidt [55, 57].

C) Having understood the special relativistic case, we are now in a position to study the Hamiltonian structure of general relativistic adiabatic fluids, by using essentially the same method again, modulo changes to include the metric tensor g_{ij} as a dynamical variable.

4. General relativistic adiabatic fluids

The aim of this section is to use the Poisson structure algorithm developed in the previous two sections to construct the Hamiltonian formalism for general relativistic adiabatic fluids in terms of the physical fluid variables and canonical gravitational variables by using the formalism of Arnowitt, Deser and Misner [1] (abbreviated as ADM).

A number of differences from the special relativistic case can be anticipated for the Hamiltonian formalism of general relativistic adiabatic fluids. In particular,

1) There are no preferred reference frames, as there are in the Minkowski case (the fluid and laboratory frames). The dynamics will be considered as taking place on arbitrary spacelike hypersurfaces.

2) The metric tensor, g_{ij}, induced on such a spacelike hypersurface will be a dynamical variable, i.e., $\partial_t g_{ij} \neq 0$.

3) Not all of the Einstein field equations will be dynamical: while the induced metric g_{ij} de-

termines the dynamics of the field, the space–time metric components $g_{\mu 0}$, $\mu = 0, 1, \ldots, n$, may be chosen arbitrarily, and the corresponding Einstein field equations give rise to constraints involving g_{ij} and $\partial_t g_{ij}$, but not involving $\partial_t^2 g_{ij}$.

Each of these differences is elucidated in the Hamiltonian framework derived below in terms of the physical fluid variables.

For reference, a table of contents for this section is provided:

4.1. *Equations of motion and notation*

In general relativity, the gravitational field and the motion of an adiabatic fluid under its own

gravity are determined self-consistently by the equations

$$G_{\mu\nu} = \frac{k}{2c^4} T_{\mu\nu}, \quad T^{\mu\nu}{}_{;\nu} = 0, \quad u^\mu \eta_{f,\mu} = 0.$$

$$(4.1)$$

Here, the gravitational coupling constant is $k = 16\pi G$, G is Newton's gravitational constant, and $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R. \tag{4.2}$$

In (4.2), the notational conventions of MTW are used: $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature; $R_{\mu\nu}$ is the Ricci tensor, defined in terms of the curvature tensor $R^\kappa{}_{\mu\lambda\nu}$ for the metric $g_{\mu\nu}$ by $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, with

$$R^\kappa{}_{\mu\lambda\nu} := \Gamma^\kappa{}_{\mu\nu,\lambda} - \Gamma^\kappa{}_{\mu\lambda,\nu} + \Gamma^\kappa{}_{\sigma\lambda}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\kappa{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\lambda}, \tag{4.3a}$$

where $\Gamma^\kappa{}_{\mu\nu}$ is the Christoffel symbol of the second kind,

$$\Gamma^\kappa{}_{\mu\nu} := \tfrac{1}{2} g^{\kappa\lambda} [g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}]. \tag{4.3b}$$

From the Bianchi identities,

$$G^{\mu\nu}{}_{;\nu} = 0, \tag{4.4}$$

and Einstein's field eq. (4.1a), there follows the fluid energy–momentum conservation law (4.1b)

$$T^{\mu\nu}{}_{;\nu} = 0,$$

where the energy-momentum tensor $T^{\mu\nu}$ of the fluid is [cf. eq. (3.3)]

$$T_{\mu\nu} = \rho_f c^2 w u_\mu u_\nu + p_f g_{\mu\nu}, \tag{4.5a}$$

$$w = 1 + (e_f + p_f/\rho_f) c^{-2}, \tag{4.5b}$$

and the timelike velocity vector u^μ of the fluid satisfies

$$g_{\mu\nu} u^\mu u^\nu = -1, \tag{4.6}$$

which is preserved by eqs. (4.1). Subscripts f in

(4.1c) and (4.5a, b) denote fluid variables as measured in the frame of an observer at rest with respect to the fluid. The other notation in (4.1), (4.5–6) is the same as that defined for special relativistic fluids, in the previous section. In particular, Greek indices run from 0 to n, Latin indices run from 1 to n, and repeated indices are summed. As in the case of special relativity, eqs. (4.1b, c) together imply the mass equation,

$$(\rho_f u^\mu)_{;\mu} = 0, \tag{4.7}$$

by projecting (4.1b) along u^μ and using the adiabaticity condition (4.1c).

4.2. ADM $(n + 1)$-decomposition of space–time

To investigate the Hamiltonian formalism for a relativistic fluid in terms of the physical fluid variables, we use the ADM notation, in which space–time is foliated into spacelike hypersurfaces depending on time parametrically. At each point of an arbitrary spacelike hypersurface $x^\alpha = X^\alpha(x^i)$ there is an $(n + 1)$-dimensional basis consisting of the n spacelike tangent vectors $X^\alpha_i := X^\alpha{}_{,i}$ and the unit timelike normal vector, n^α, such that

$$g_{\alpha\beta} X^\alpha_i n^\beta = 0 \quad \text{and} \quad g_{\alpha\beta} n^\alpha n^\beta = -1. \tag{4.8}$$

Continuous deformation of the hypersurface through space–time produces a one-parameter family of hypersurfaces $X^\alpha(x^i, t)$. The deformation vector $N^\alpha := \partial X^\alpha(x^i, t)/\partial(ct)$ connecting points with the same label x^i on two neighboring hypersurfaces can be uniquely decomposed with respect to the basis vectors $\{n^\alpha, X^\alpha_i\}$ as

$$N^\alpha = N n^\alpha + N^i X^\alpha_i. \tag{4.9}$$

The components N and N^i are called the lapse and shift functions, respectively.

Any vector in space–time can be decomposed into components perpendicular and parallel to the hypersurface. For example, the velocity $u^\alpha = (u^0, u^k)$ in space–time decomposes into $u_\perp$ and

$u_{\|}^k$ defined by

$$u^\alpha = u_\perp n^\alpha + u_{\|}^k X_k^\alpha, \tag{4.10}$$

where, using the normalization (4.8),

$$u_\perp = n_\alpha u^\alpha =: -u^\perp. \tag{4.11}$$

Solving (4.9) for n^α gives

$$n^\alpha \partial_\alpha = N^{-1} \left(\partial_{ct} - N^i \partial_i \right). \tag{4.9'}$$

Consequently, by substituting (4.9') into (4.10),

$$u^\alpha \partial_\alpha = u^\perp n^\alpha \partial_\alpha + u_{\|}^k \partial_k \tag{4.10'a}$$

$$= \left(u^\perp N^{-1} \right) \partial_{ct} + \left(u_{\|}^k - u^\perp N^{-1} N^k \right) \partial_k \tag{4.10'b}$$

$$= u^0 \partial_{ct} + u^k \partial_k, \tag{4.10'c}$$

where we identify space–time vectors with differential operators via $v^\mu \to v^\mu \partial_\mu$. Now $u^\perp$ and $u_{\|}^k$ can be expressed in terms of the lapse, shift, and space–time vector components u^α by identifying coefficients of ∂_{ct} and ∂_k in (4.10'b,c):

$$u^\perp = u^0 N, \tag{4.12a}$$

$$u_{\|}^k = u^k + u^0 N^k. \tag{4.12b}$$

In either the special relativistic, or the asymptotically flat limit, we have $N \equiv 1$, $N^k \equiv 0$, and the vector $(u^\perp, u_{\|}^k)$ reduces again to the space–time vector (u^0, u^k).

A similar decomposition exists for tensors. For the space–time metric tensor $g_{\alpha\beta}$ we already have in (4.8) the projections, $g_{\perp i} = 0$ and $g_{\perp \perp} = -1$. In addition, the tangential projection of $g_{\alpha\beta}$ along the hypersurface is

$$g_{ij}(x, t) = g_{\alpha\beta}(X(x, t)) X_i^\alpha X_j^\beta, \tag{4.13}$$

which is the n-dimensional Riemannian metric induced on the hypersurface by its embedding in space–time. As indicated in (4.13), the induced metric g_{ij} changes with time as the hypersurface is deformed. Thus, in this formalism g_{ij} is a dynamical variable.

The ADM formalism splits the metric $g_{\alpha\beta}$ into N, N^i, and g_{ij}, as follows:

$$g_{\alpha\beta} = \begin{pmatrix} N_a N^a - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}, \tag{4.14a}$$

$$g^{\alpha\beta} = \begin{pmatrix} -N^{-2} & N^{-2} N^j \\ N^{-2} N^i & g^{ij} - N^{-2} N^i N^j \end{pmatrix}, \tag{4.14b}$$

where the induced metric g_{ij} on the spacelike hypersurface raises and lowers Latin indices by

$$g_{ij} g^{jk} = \delta_i^k, \quad N_i = g_{ij} N^j, \quad N^j = g^{jk} N_k. \tag{4.15}$$

Note that from (4.14a)

$$g_{k\nu} u^\nu = N_k u^0 + u_k = g_{kl} u_{\|}^l =: u_k^{\|} \quad \text{[by 4.12b]}. \tag{4.16a}$$

The ADM decomposition then allows the proof of useful relations among space–time components of $(n+1)$-decomposed components of tensors and vectors. One such relation is

$$\sqrt{-^{(n+1)}g} := \sqrt{-\det(g_{\alpha\beta})} = N\sqrt{\det(g_{ij})} =: N\sqrt{g}. \tag{4.16b}$$

So, the volume element has the form

$$\sqrt{-^{(n+1)}g} \, dx^0 dx^1 \ldots dx^n = cN\sqrt{g} \, dt \, dx^1 \ldots dx^n. \tag{4.16c}$$

The rate of change of g_{ij} with respect to the time label, t, can also be decomposed into normal and tangential components (see, e.g., MTW, p. 513)

$$\partial_t g_{ij} = -2N K_{ij} + N_{i|j} + N_{j|i}, \tag{4.17}$$

where the vertical stroke $|$ denotes spatial covariant derivative with respect to the induced metric g_{ij}, and $K_{ij} = -n_{i;j} = -N\Gamma_{ij}^0$ is the extrinsic curvature of the hypersurface. Eq. (4.17) may be rewritten as

$$\partial_t g_{ij} = -2N K_{ij} + \mathscr{L}_N g_{ij}, \tag{4.18}$$

with $\mathscr{L}_N g_{ij}$ being the Lie derivative of g_{ij} with respect to $N := N^i \partial_i$. Then, using (4.9') the extrinsic curvature tensor is given by (cf. MTW, p. 518)

$$K_{ij} = -\tfrac{1}{2} \mathscr{L}_n g_{ij}, \tag{4.19}$$

where $n := n^\alpha \partial_\alpha$. Finally, we can decompose the energy–momentum tensor as

$$T^{\perp\perp} = \rho_f c^2 w (u^\perp)^2 - p_f$$
$$= -T^\perp_{\;\perp} \quad [\text{since } g_{\perp\perp} = -1], \tag{4.20a}$$

$$T^{\perp\parallel}_k = \rho_f c^2 w u^\perp u^\parallel_k =: cM_k/\sqrt{g}, \tag{4.20b}$$

$$T^{\parallel\parallel}_{ij} = \rho_f c^2 w u^\parallel_i u^\parallel_j + g_{ij} p_f, \tag{4.20c}$$

so that the Einstein equation (4.1a) decomposes into

$$\frac{1}{2\sqrt{g}} \mathscr{H} := G^\perp_{\;\perp} - \frac{1}{2} \frac{k}{c^4} T^\perp_{\;\perp} = 0, \tag{4.21a}$$

$$\frac{1}{2\sqrt{g}\,c^2} \mathscr{J}_i := G^{\perp\parallel}_i - \frac{1}{2} \frac{k}{c^4} T^{\perp\parallel}_i = 0, \tag{4.21b}$$

$$G^{i\parallel}_{\parallel j} - \frac{k}{2c^4} T^{i\parallel}_{\parallel j} = 0, \tag{4.21c}$$

with the perpendicular and parallel components of the Einstein tensor given, e.g., in MTW (21.162a–c), p. 552. The symbols $\mathscr{H}$ and $\mathscr{J}_i$ defined in (4.21a–c) will reappear momentarily in the Hamiltonian density for the general relativistic adiabatic fluid. For now, we note that MTW (21.162a–c), p. 552, becomes upon including the various factors of c and using normalization (4.8):

$$G^\perp_{\;\perp} = -\frac{1}{2} {}^{(n)}R + \frac{1}{2c^2} \left[K_{ij} K^{ij} - (K^i_i)^2 \right], \tag{4.22a}$$

$$G^{\perp\parallel}_i = \frac{1}{c^2} \left[K^m_{i|m} - K^m_{m|i} \right], \tag{4.22b}$$

$$G^{i\parallel}_{\parallel j} = {}^{(n)}G^i_j - \left\{ c^{-1} N^{-1} \left(\partial_{ct} - N^k \partial_k \right) \left(K^i_j - \delta^i_j K^m_m \right) \right.$$
$$- c^{-2} K^m_m K^i_j + \tfrac{1}{2} c^{-2} \delta^i_j (K^m_m)^2$$
$$\left. + \tfrac{1}{2} c^{-2} \delta^i_j K_{lm} K^{lm} \right\}, \tag{4.22c}$$

where ${}^{(n)}R$ is the scalar curvature and ${}^{(n)}G^i_j$ is the Einstein tensor formed from the induced metric g_{ij}.

4.3. Action principle and Legendre transformation to canonical variables

Arnowitt, Deser, and Misner [1] showed that the Hilbert action,

$$S_H[g_{\alpha\beta}] = \int dx^0 d^n x \, (2ck)^{-1} \sqrt{-{}^{(n+1)}g} \, R,$$

could be written (modulo divergences: we are ignoring boundary terms) in terms of the field variables g_{ij}, N, N^j, and the extrinsic curvature K_{ij} as

$$S_{ADM}[g_{ij}, N, N^j] = \int dt \int d^n x \left\{ (2k)^{-1} N \sqrt{g} \left[c^{-2} K_{ij} K^{ij} - c^{-2} (K^i_i)^2 + {}^{(n)}R \right] \right\}, \tag{4.23}$$

where ${}^{(n)}R$ is the scalar curvature formed from the induced metric g_{ij}. From the action (4.23), ADM obtained the canonical equations for the field equations (3.1a) with $T_{\mu\nu} = 0$, i.e., in the absence of matter.

Recently, Künzle and Nester [8] (abbreviated as KN) used the ADM formalism to obtain a canonical Hamiltonian form of both the field equations (4.1a) and the fluid dynamics of a barotropic $[p_f = \bar{p}(\rho_f)]$, general relativistic fluid (4.1b) from the sum of the ADM field action (4.23) and a fluid action reminiscent of (3.11). For an *adiabatic* general relativistic fluid, the KN action becomes

$$L_{KN} = L_{ADM} - \int dt \int d^n x \, N \sqrt{g} \, \varepsilon(\rho_f, \eta_f), \tag{4.24}$$

with ρ_f and η_f defined in the space of Lagrangian variables, $q^A(x, t)$, as in the previous sections.

We adopt the action (4.24) as the starting point for deriving by means of the Poisson structure algorithm of sections 2 and 3 the noncanonical Poisson structure of a gravitating fluid found empirically in BMW [12]. To do this, we first rewrite (4.1b, c) and (4.7) in the ADM decomposition as

$$\partial_t \hat{\rho} = -(\hat{\rho} v^i)_{,i},$$ (4.25a)

$$\partial_t \eta = -v^i \eta_{,i},$$ (4.25b)

$$\partial_t s_i = -v^j s_{i|j} - \frac{N\sqrt{g}}{\hat{\rho}} p_{f,i} - c^2 w u^{\perp} N_{,i} + c s_k N^k_{|i},$$ (4.25c)

where, as before, the vertical stroke $|$ represents covariant derivative in the spacelike hypersurface with lapse N and shift N^k, and the following additional notation is introduced:

$$v^k := c u^k / u^0 \ [\text{by } (4.12)] = c N \left(\frac{u^k_{\|}}{u^{\perp}} - \frac{N^k}{N} \right),$$ (4.26a)

$$s_i := c w u^{\|}_i,$$ (4.26b)

$$\hat{\rho} := \rho_f u^0 N \sqrt{g} \ [\text{by } (4.12a)] = \rho_f u^{\perp} \sqrt{g},$$ (4.26c)

$$\eta := \eta_f.$$ (4.26d)

The fluid equations of motion (4.25a–c) are obtained from the starting equations (4.1b, c) and their implied eq. (4.7), as follows. From the mass equation (4.7) we have

$$0 = \left(\sqrt{-^{(n+1)}g} \right) \left(c \sqrt{-^{(n+1)}g} \, \rho_f u^{\mu} \right)_{;\mu} = c \left(\rho_f u^0 \sqrt{-^{(n+1)}g} \right)_{,ct} + c \left(\rho_f u^i \sqrt{-^{(n+1)}g} \right)_{,i}$$

$$[\text{by } (4.16b)] = \left(\rho_f u^0 N \sqrt{g} \right)_{,t} + \left(\rho_f u^0 N \sqrt{g} \, c u^i / u^0 \right)_{,i}$$

$$[\text{by } (4.26c)] = \hat{\rho}_{,t} + \left(\hat{\rho} c u^i / u^0 \right)_{,i}$$

$$[\text{by } (4.26a)] = \hat{\rho}_{,t} + \left(\hat{\rho} v^i \right)_{,i}.$$

This proves (4.25a). Next, from the entropy eq. (4.1c), there follows

$$0 = u^{\mu} \eta_{,\mu} = u^0 \eta_{,ct} + u^k \eta_{,k} = \frac{u^0}{c} \left[\eta_{,t} + \frac{c u^k}{u^0} \eta_{,k} \right]$$

$$[\text{by } (4.26a)] = \frac{u^0}{c} \left[\eta_{,t} + v^k \eta_{,k} \right].$$

This proves (4.25b). Finally, the motion equation (4.25c) is derived in appendix A, by taking the tangential ith component of the covariant fluid equation (4.1b) in the ADM decomposition.

When $N = 1$ and $N^k = 0$, the motion equation (4.25c) reduces to the special relativistic fluid motion equation (3.9c). The other equations (4.25a, b), are just the same, respectively, as eqs. (3.9a, b) in the case of special relativity. Thus, the variables $\hat{\rho}$, η, and v^i can be defined in terms of Lagrangian coordinates $q^A(x, t)$, $A = 1, 2, \ldots, n$, in the same way as before, in (3.10a–d). Namely,

$$\hat{\rho}(x, t) = \bar{\rho}(q) \det\left(q^A_i \right), \quad \eta(x, t) = \bar{\eta}(q), \quad \dot{q}^A + q^A_j v^j = 0, \quad v^j = -\left(q^{-1} \right)^j_A \dot{q}^A,$$ (3.10a, b, c, d)

where, as before, $q_i^A = \partial q^A / \partial x^i$, $\dot{q}^A = \partial q^A / \partial t$, $(q^{-1})_A^j$ is the matrix inverse of q_j^A, and $\bar{\rho}$ and $\bar{\eta}$ are prescribed functions of the entire set of Lagrange coordinates $q := \{q^A\}$.

The spatial fluid velocity v^j is defined in terms of Lagrangian coordinates by eq. (3.10d). The fluid velocity appropriate for the action principle (4.24) is not v^j in (3.10d), though. Rather, the appropriate velocity is t^k, the velocity tangential to the spacelike hypersurface. This velocity is defined by

$$\frac{t^k}{c} := \frac{u_\parallel^k}{u^\perp} \text{ [by (4.12a,b)]} = \frac{u^k}{u^\perp} + \frac{N^k}{N}. \tag{4.27}$$

Now, according to (4.26a), the velocities v^k and t^k are related, since

$$\frac{v^k}{c} = N\left(\frac{u_\parallel^k}{u^\perp} - \frac{N^k}{N}\right), \tag{4.28a}$$

$$\text{[by (4.27)]} = N\left(\frac{t^k}{c} - \frac{N^k}{N}\right). \tag{4.28b}$$

Then, as a consequence of (3.10d) and (4.28b), the tangential velocity t^k is related to the Lagrangian velocity $\dot{q}^A$ by

$$t^k = -N^{-1}\left(\dot{q}^A - cN^j q_j^A\right)(q^{-1})_A^k, \tag{4.29a}$$

so that

$$\frac{\partial t^k}{\partial \dot{q}^A} = -N^{-1}(q^{-1})_A^k. \tag{4.29b}$$

The velocity t^k also appears in $u^\perp$, since

$$u^\perp = \left(1 - |t|^2/c^2\right)^{-1/2}, \tag{4.30}$$

where $|t|^2 := g_{ij}t^i t^j$. Relation (4.30) can be seen by using the ADM representation (4.14a) of the metric $g_{\alpha\beta}$ in the normalization condition (4.6):

$$-1 = g_{\alpha\beta}u^\alpha u^\beta = -(u^\perp)^2 + |u_\parallel|^2 = -(u^\perp)^2\left[1 - |u_\parallel|^2/(u^\perp)^2\right] = -(u^\perp)^2(1 - |t|^2/c^2), \tag{4.31}$$

where $|u_\parallel|^2 := g_{ij}u_\parallel^i u_\parallel^j$. Eq. (4.30) then follows from (4.31) by solving for $u^\perp$. The derivative of (4.30) with respect to t^k gives

$$\frac{\partial (u^\perp)^{-1}}{\partial t^k} = -\frac{u^\perp t_k}{c^2} = -\frac{u_k^\parallel}{c}, \tag{4.32}$$

where $t_k = g_{kj}t^j$. The derivative relation (4.32) will be useful momentarily in taking the variational derivative with respect to $\dot{q}^A$ of the KN action (4.24), in order to obtain the canonical momentum.

The canonical momentum variables are obtained from the action (4.24), as follows. Varying action (4.24) with respect to $\dot{g}_{ij} := \partial_t g_{ij}$ defines the canonical field momentum density π^{ij} [KN, eq. (3.18)] as

$$k^{-1}c^2\pi^{ij} := \frac{\delta L_{KN}}{\delta \dot{g}_{ij}} = k^{-1}c^2\sqrt{g}\left(g^{ij}K_r^r - K^{ij}\right). \tag{4.33}$$

Varying L_{KN} in (4.24) with respect to $\dot{q}^A$ gives the canonical fluid momentum density

$$P_A := \frac{\delta L_{KN}}{\delta \dot{q}^A} = -N\sqrt{g}\,\frac{\partial \varepsilon}{\partial \rho_f}\,\frac{\partial \rho_f}{\partial (u^\perp)^{-1}}\,\frac{\partial (u^\perp)^{-1}}{\partial t^k}\,\frac{\partial t^k}{\partial \dot{q}^A} = -N\sqrt{g}\,(c^2 w)\left(\frac{\hat{\rho}}{\sqrt{g}}\right)\left(-\frac{u_k^\parallel}{c}\right)\left(-N^{-1}(q^{-1})^k_A\right)$$

$$= -cw\hat{\rho}u_k^\parallel(q^{-1})^k_A \;[\text{by (4.26b)}] = -\hat{\rho}s_k(q^{-1})^k_A =: -\bar{\theta}t_k(q^{-1})^k_A. \tag{4.34}$$

In the first line of the computation (4.34) we have used (3.13), (4.26c), (4.29b), and (4.32). In the second line we have used (4.27) and introduced the quantity $\bar{\theta}$,

$$\bar{\theta} := \hat{\rho}u^\perp w, \tag{4.35}$$

where according to (4.26c) $\hat{\rho} := \rho_f u^0 N\sqrt{g} = \rho_f u^\perp \sqrt{g}$.

The physical fluid momentum density M_k is defined to be

$$M_k \;[\text{by (4.20b)}] := c^{-1}N\sqrt{g}\,T_k^0 \;[\text{by (4.16)}] = cw\hat{\rho}u_k^\parallel \;[\text{by (4.26b) and (4.34)}]$$

$$= \hat{\rho}s_k \;[\text{by (4.34)}] = \bar{\theta}t_k \;[\text{by (4.34)}] = -P_A q_k^A. \tag{4.36}$$

Remark. The variable M_k in (4.36) is related to the momentum density variable μ in BMW [32] by $M_k = \frac{1}{2}\mu_k$.

4.4. Canonical Hamiltonian formulation

Relations (2.10a,b) and (4.36) will soon enable us to map from the canonical Poisson structure for the variables (P_A, q^A) to a noncanonical Poisson structure for the fluid variables $\{\hat{\rho}, \eta, M_k\}$. (As before, all these variables are functions of Eulerian coordinates.) Before making this transition in the Poisson structure, though, we present a slightly modified version of the canonical Hamiltonian formalism for gravitating fluids discussed by KN, in order to obtain the gravitational "super-Hamiltonian". Rewriting the last relation in (4.34) and using (4.32), we obtain

$$P_A = -\bar{\theta}t_k(q^{-1})^k_A = N^{-1}\bar{\theta}(\dot{q}^B - cN^l q_l^B)(q^{-1})^m_B g_{mk}(g^{-1})^k_A = N^{-1}\bar{\theta}(\Delta^{-1})_{AB}(\dot{q}^B - cN^l q_l^B), \tag{4.37}$$

where $(\Delta^{-1})_{AB}$ is defined in (2.9). Hence,

$$\Delta^{AB}P_B = N^{-1}\bar{\theta}(\dot{q}^A - cN^l q_l^A) \tag{4.38}$$

and we find the following relations:

$$\frac{N}{\bar{\theta}}|P|^2 := \frac{N}{\bar{\theta}}P_A\Delta^{AB}P_B \;[\text{by (4.38)}] = P_A(\dot{q}^A - cN^l q_l) \;[\text{by (4.32')}]$$

$$= -Nt^k P_A q_k^A \;[\text{by (4.36)}] = Nt^k M_k = N\bar{\theta}|t|^2 = \frac{N}{\bar{\theta}}|M|^2. \tag{4.39}$$

Thus, the momentum density magnitudes are equal, $|P|^2 = |M|^2$. Using (4.38), (4.39), and (4.36), gives

$$P_A \dot{q}^A = \frac{N}{\theta}|P|^2 + cN^l P_A q_l^A \quad \text{[by (4.39)]} \tag{4.40a}$$

$$= \frac{N}{\theta}|M|^2 - cN^l M_l. \tag{4.40b}$$

Thus, the fluid Hamiltonian obtained by Legendre transforming L_{KN} in (4.24) will be easily expressible in terms of either Lagrangian variables (P_A, q^A), or Eulerian variables $\{\hat{\rho}, \eta, M_k\}$. In either case, the lapse and shift functions will appear explicitly and only linearly. Note that the time derivatives of N and N^i are absent from both the field (ADM) piece of the action and the fluid piece (for further discussion of this phenomenon, see Gimmsy [58]).

The lapse and shift functions that define the ADM decomposition are merely prescribed functions, not dynamical variables.

The KN action (4.24) is expressible as a phase space functional, parametrized by N, N^a,

$$L_{\mathrm{KN}}\left[\pi^{ab}, g_{ab}, P_A, q^A; N, N^a\right] = \left(k^{-1}c^2\right)\int \mathrm{d}t \int \mathrm{d}^n x \left[\pi^{ab}\dot{g}_{ab} + kc^{-2}P_A \dot{q}^A - c^2 N\mathcal{H} - N^a \mathcal{J}_a\right], \tag{4.41}$$

where π^{ij} is defined in (4.33), P_A is defined in (4.37), and $\mathcal{H}$ is the so-called super-Hamiltonian density

$$\mathcal{H} := \mathcal{H}_{\mathrm{gr}} + \mathcal{H}_{\mathrm{fl}}, \tag{4.42}$$

with gravitational part

$$\mathcal{H}_{\mathrm{gr}} := -\sqrt{g}\,{}^{(n)}R + c^{-2}\left(\sqrt{g}\right)^{-1}\left(\pi^{ab}\pi_{ab} - \frac{1}{(n-1)}(\pi_r^r)^2\right) \tag{4.43}$$

and fluid part

$$\mathcal{H}_{\mathrm{fl}} := \mathcal{H}_{\mathrm{fl}}\left(q^A, q_i^A, g_{ab}, P_A\right) := kc^{-4}\left(\sqrt{g}\,\varepsilon(\rho_f, \eta_f) + |P|^2/\theta\right) \tag{4.44a}$$

$$\text{[by (4.39)]} = kc^{-4}\left(\sqrt{g}\,\varepsilon(\rho_f, \eta_f) + |M|^2/\theta\right). \tag{4.44b}$$

The supermomentum density $\mathcal{J}_a$ in (4.41) is defined as

$$\mathcal{J}_a := \mathcal{J}_a^{\mathrm{gr}} + \mathcal{J}_a^{\mathrm{fl}} := -2\pi_{a|r}^r + \left(kc^{-2}\right)cP_A q_a^A \tag{4.45a}$$

$$\text{[by (3.36)]} = -2\pi_{a|r}^r - kc^{-1}M_a. \tag{4.45b}$$

The Hamiltonian for the combined system of fluid and field is now written as

$$H = \left(k^{-1}c^2\right)\int \mathrm{d}^n x \left[Nc^2(\mathcal{H}_{\mathrm{gr}} + \mathcal{H}_{\mathrm{fl}}) + N^a(\mathcal{J}_a^{\mathrm{gr}} + \mathcal{J}_a^{\mathrm{fl}})\right],$$

$$H =: H_{\mathrm{gr}} + H_{\mathrm{fl}}, \tag{4.46}$$

using definitions (4.43)–(4.45). Note, when the metric g_{ij} in (4.46) is time independent, and $N \equiv 1$, $N^a \equiv 0$, then H returns to the special relativistic form (3.23).

The physical meaning of the field quantities $\mathcal{H}_{gr}$ in (4.43) and $\mathcal{J}_a^{gr}$ in (4.45) can be obtained readily by substituting the definition of π^{ij} (4.33) into (4.43) and (4.45), and comparing the results with (3.22a, b), whereupon we have

$$\mathcal{H}_{gr} = 2\sqrt{g}\, G^{\perp}{}_{\perp}, \tag{4.47a}$$

$$\mathcal{J}_a^{gr} = -2\pi_{a|m}^{m} = 2\sqrt{g}\, c^2 G^{\perp\|}{}_{a}. \tag{4.47b}$$

Thus, $\mathcal{H}_{gr}$ and $\mathcal{J}_a^{gr}$ are proportional to the $\perp\perp$ and $\perp\|$ components of the Einstein tensor. This is, of course, well known, see, e.g., Isham [5].

The physical meaning of the fluid quantities $\mathcal{H}_{fl}$ in (4.44b) and $\mathcal{J}_a^{fl}$ in (4.46b) can be elucidated in terms of the fluid stress tensor, $T^{\mu\nu}$, from eq. (4.5),

$$T^{\mu\nu} = \rho_f c^2 w u^\mu u^\nu + {}^{(n+1)}g^{\mu\nu} p_f. \tag{4.48}$$

Namely, $\mathcal{H}_{fl}$ and $\mathcal{J}_a^{fl}$ are related to $T^{\mu\nu}$ by

$$c^4 k^{-1}\mathcal{H}_{fl} = \sqrt{g}\, T^{\perp}{}_{\perp} = c^2\bar{\theta} - \sqrt{g}\, p_f, \tag{4.49a}$$

$$\mathcal{J}_a^{fl} = -\sqrt{g}\, N(kc^{-2})T_a^0 = -\sqrt{g}\,(kc^{-2})T^{\perp\|}{}_a. \tag{4.49b}$$

These relations can be shown directly. By (4.47b) and (4.36), we have (4.49b),

$$\mathcal{J}_a^{fl} = -kc^{-1}M_a = -kc^{-2}N\sqrt{g}\, T_a^0 = -kc^{-2}\sqrt{g}\, T^{\perp\|}{}_a.$$

For (4.49a), we write

$$\sqrt{g}\, T^{\perp}{}_{\perp} := N^2\sqrt{g}\, T^{00} \;\left[\text{by (4.48)}\right] = N^2\sqrt{g}\left(\rho_f c^2 w u^0 u^0 + {}^{(n+1)}g^{00}p_f\right)$$

$$\left[\text{by (4.14b)}\right] = N^2\sqrt{g}\left(\rho_f c^2 w(u^0)^2 - N^{-2}p_f\right)$$

$$\left[\text{by (4.12a)}\right] = \sqrt{g}\left(\rho_f c^2 w(u^{\perp})^2 - p_f\right)\;\left[\text{by (4.26b) and (4.35)}\right] = c^2\bar{\theta} - \sqrt{g}\, p_f$$

$$= \sqrt{g}\left[\rho_f c^2 w\left((u^{\perp})^2 - 1\right) + \rho_f c^2 w - p_f\right]$$

$$\left[\text{by (3.13) and (4.29)}\right] = \sqrt{g}\left[\rho_f c^2 w|\boldsymbol{u}_{\|}|^2 + \varepsilon(\rho_f, \eta_f)\right]$$

$$\left[\text{by (4.36) and (4.26b)}\right] = \frac{|M|^2}{u^{\perp}w\hat{\rho}} + \sqrt{g}\,\varepsilon(\rho_f, \eta_f)$$

$$\left[\text{by (4.35)}\right] = |M|^2/\bar{\theta} + \sqrt{g}\,\varepsilon(\rho_f, \eta_f)\;\left[\text{by (4.44b)}\right] = c^4 k^{-1}\mathcal{H}_{fl}.$$

In this computation, equating the first expression with the last one and with the last expression on the third line proves relations (4.49a).

Upon combining (4.47) and (4.49), we regain (4.21a, b), namely

$$\mathcal{H} = \mathcal{H}_{gr} + \mathcal{H}_{fl} = 2\sqrt{g}\left(G^{\perp}{}_{\perp} - \frac{k}{2c^4}T^{\perp}{}_{\perp}\right), \tag{4.21a}$$

$$\mathcal{J}_a = \mathcal{J}_a^{gr} + \mathcal{J}_a^{fl} = 2\sqrt{g}\, c^2\left(G^{\perp\|}{}_a - \frac{k}{2c^4}T^{\perp\|}{}_a\right). \tag{4.21b}$$

So the Hamiltonian (4.46) may be rewritten as

$$H = 2k^{-1}c^4 \int d^n x \left[\sqrt{g} \, N \left(G^\perp{}_\perp - \frac{k}{2c^4} T^\perp{}_\perp \right) + \sqrt{g} \, N^a \left(G^{\perp\|}{}_a - \frac{k}{2c^4} T^{\perp\|}{}_a \right) \right]. \tag{4.46'}$$

Thus, the Hamiltonian H in (4.46') contains the $\perp \perp$ and $\perp \|$ components of Einstein's equations, in a linear combination with (in general) time-dependent coefficients.

4.5. Map from canonical to physical fluid variables

As in the case of special relativistic adiabatic fluids, the Hamiltonian (4.46) depends implicitly via relations (4.49a, b) on the field and fluid variables through the quantity $\bar{\theta}$ defined in (4.35). This implicit dependence will cause no difficulty, however, in passing from the canonically conjugate set of variables $\{ g_{ij}, \pi^{ij}, P_A, q^A \}$ to the physical variables $\{ g_{ij}, \pi^{ij}, \hat{\rho}, \eta, M_i \}$. Before this transition, though, we first replace P_A by $(-P_A)$ in the Hamiltonian (4.47) and in eq. (4.36). This replacement is immaterial in the Hamiltonian, but it conveniently changes the last sign in (4.36), thus [along with (3.10a, b)] reproducing the map (2.15) from the previous sections, but now in terms of general relativistic variables. Namely,

$$M_i = P_A \partial_i q^A, \quad \hat{\rho} = \bar{\rho}(q) \det (q_i^A), \quad \eta = \bar{\eta}(q), \tag{4.50}$$

where M_i, $\hat{\rho}$, and η are given, respectively, by eqs. (4.36), (4.26a), and (4.26b). We know that this map is canonical, since it yields the Lie–Poisson bracket (2.18) in terms of $\{ \hat{\rho}, \eta, M_i \}$ which is associated to the semidirect-product Lie algebra discussed at the end of section 2. The Hamiltonian functional (4.46) can be expressed in terms of the functions N and N^i, and the set $\{ g_{ij}, \pi^{ij}, \hat{\rho}, \eta, M_i \}$ only, through eqs. (4.43), (4.44b), (4.45b). In particular, the fluid piece of the Hamiltonian (4.47) is

$$H_{\mathrm{fl}} = (k^{-1}c^2) \int d^n x \left[Nc^2 \mathcal{H}_{\mathrm{fl}} + N^a \mathcal{J}_a^{\mathrm{fl}} \right]$$

$$[\text{by (4.44b) and (4.46b)}] = (k^{-1}c^2) \int d^n x \left[Nkc^{-2} \left(\sqrt{g} \, \varepsilon(\rho_f, \eta) + |M|^2/\bar{\theta} \right) - kc^{-1} N^a M_a \right] \tag{4.51}$$

$$[\text{by (4.49a)}] = \int d^n x \left[N \left(c^2 \bar{\theta} - \sqrt{g} \, p_f \right) - c N^a M_a \right],$$

which is the general relativistic analog of (3.25).

The variational derivatives of the fluid Hamiltonian (4.51) needed for the equations of motion are

$$\frac{\delta H_{\mathrm{fl}}}{\delta M_i} = v^i, \quad \frac{\delta H_{\mathrm{fl}}}{\delta \hat{\rho}} = \frac{Nc^2 w}{u^\perp}, \quad \frac{\delta H_{\mathrm{fl}}}{\delta \eta} = N \sqrt{g} \, \rho_f T_f, \quad \frac{\delta H_{\mathrm{fl}}}{\delta \pi^{ij}} = 0, \quad \frac{\delta H_{\mathrm{fl}}}{\delta g_{ij}} = -\tfrac{1}{2} \sqrt{g} \, N T^{ij}_{\|\|}. \tag{4.52a–e}$$

To prove variational identities (4.52a–e), one first needs that

$$\hat{\rho} u^\perp w =: \bar{\theta} = \sqrt{|M|^2/c^2 + (\hat{\rho}w)^2}, \tag{4.53}$$

which follows by substituting $u^\perp$ from (4.35) into (4.31), and using (4.39). We also need (3.27), rewritten as

$$c^2 dw = \rho_f^{-1} \frac{\partial p_f}{\partial \rho_f} d\rho_f + \left(\rho_f^{-1} \frac{\partial p_f}{\partial \eta_f} + T_f \right) d\eta_f, \quad dp_f = \rho_f c^2 dw - \rho_f T_f d\eta_f. \tag{4.54}$$

By using (4.53), (4.54), and (4.26b, c), we obtain from (4.51)

$$\frac{\delta H_{\mathrm{fl}}}{\delta M_i} = N\left(c^2\frac{\partial\bar{\theta}}{\delta M_i} - \sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial M_i}\right) - cN^i = N\frac{c^2}{\bar{\theta}}\left(M^i/c^2 + \hat{\rho}^2 w\frac{\partial w}{\partial\rho_{\mathrm{f}}}\frac{\partial\rho_{\mathrm{f}}}{\partial M_i}\right) - N\sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\rho_{\mathrm{f}}}\frac{\partial\rho_{\mathrm{f}}}{\partial M_i} - cN^i$$

$$= N\frac{M^i}{\bar{\theta}} - cN^i + N\frac{\partial\rho_{\mathrm{f}}}{\partial M_i}\left(\frac{c^2\hat{\rho}}{u^{\perp}}\frac{\partial w}{\partial\rho_{\mathrm{f}}} - \sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\rho_{\mathrm{f}}}\right) = N\frac{M^i}{\bar{\theta}} - cN^i = Nt^i - cN^i = v^i, \qquad (4.55)$$

which proves (4.52a). Next, by (4.53) and (4.54)

$$\frac{\delta H_{\mathrm{fl}}}{\delta\hat{\rho}} = \frac{Nc^2}{\bar{\theta}}\hat{\rho}w\frac{\partial(\hat{\rho}w)}{\partial\hat{\rho}} - \sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\hat{\rho}} = \frac{Nc^2}{u^{\perp}}\left(w + \hat{\rho}\frac{\partial w}{\partial\hat{\rho}}\right) - \sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\hat{\rho}}$$

$$= \frac{Nc^2 w}{u^{\perp}} + \sqrt{g}\left(c^2\rho_{\mathrm{f}}\frac{\partial w}{\partial\rho_{\mathrm{f}}} - \frac{\partial p_{\mathrm{f}}}{\partial\rho_{\mathrm{f}}}\right)\frac{\partial\rho_{\mathrm{f}}}{\partial\hat{\rho}} = \frac{Nc^2 w}{u^{\perp}}, \qquad (4.56)$$

which is (4.52b). Now, by (4.53) and (4.54) again,

$$\frac{\delta H_{\mathrm{fl}}}{\partial\eta} = \frac{Nc^2}{\bar{\theta}}\hat{\rho}w\frac{\partial\hat{\rho}w}{\partial\eta} - N\sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\eta} = \frac{Nc^2}{u^{\perp}}\hat{\rho}\frac{\partial w}{\partial\eta} - N\sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial\eta}$$

$$= N\sqrt{g}\left(\rho_{\mathrm{f}}c^2\frac{\partial w}{\partial\eta} - \frac{\partial p_{\mathrm{f}}}{\partial\eta}\right) = N\sqrt{g}\,\rho_{\mathrm{f}}T_{\mathrm{f}}, \qquad (4.57)$$

which is (4.52c). The variational derivative (4.52d) is clear, since H_{fl} does not contain π^{ij}. Finally, (4.52e) is obtained as follows:

$$\frac{\delta H_{\mathrm{fl}}}{\delta g_{ij}} = Nc^2\frac{\partial\bar{\theta}}{\partial g_{ij}} - Np_{\mathrm{f}}\frac{\partial\sqrt{g}}{\partial g_{ij}} - N\sqrt{g}\,\frac{\partial p_{\mathrm{f}}}{\partial g_{ij}}$$

$$= -\frac{N}{2}\left[\frac{M^i M^j}{\bar{\theta}} + p_{\mathrm{f}}\sqrt{g}\,g^{ij}\right] + N\left[\frac{\hat{\rho}^2 w}{\bar{\theta}} - \sqrt{g}\,\rho_{\mathrm{f}}\right]\frac{\partial c^2 w}{\partial g_{ij}}, \qquad (4.58)$$

where we have used the definition of $\bar{\theta}$ (4.35), the thermodynamic relation (4.54b), and the identity

$$\frac{\partial\sqrt{g}}{\partial g_{ij}} = \tfrac{1}{2}\sqrt{g}\,g^{ij}.$$

Using the definitions of $\bar{\theta}$ and $\hat{\rho}$ in the relation (4.59) now gives

$$\frac{\delta H_{\mathrm{fl}}}{\delta g_{ij}} = -\tfrac{1}{2}N\sqrt{g}\left[\frac{M^i M^j}{\sqrt{g}\,\bar{\theta}} + p_{\mathrm{f}}g^{ij}\right]\,[\text{by }(3.20\mathrm{b,c})] = -\tfrac{1}{2}N\sqrt{g}\,T_{\mathrm{fl}}^{ij}.$$

This proves (4.52e).

4.6. *Hamiltonian formulation in terms of physical fluid variables*

Substituting the variational derivatives (4.52a–c) into the Lie–Poisson bracket (2.18) and noting that H_{gr} depends only on the canonical fields (g_{ij}, π^{ij}) readily yields

$$\partial_t \hat\rho = \{H, \hat\rho\} = -(\hat\rho v^i)_{,i}, \qquad (4.59a)$$

$$\partial_t \eta = \{H, \eta\} = -\eta_{,i} v^i, \qquad (4.59b)$$

which recover eqs. (4.25a) and (4.25b), respectively, in the Hamiltonian formalism, where $H = H_{\mathrm{gr}} + H_{\mathrm{fl}}$ is the sum of the gravitational and fluid Hamiltonians.

Substituting variational derivatives (4.52a–c) into the Lie–Poisson bracket (2.18) also gives

$$\partial_t M_i = \{H, M_i\}$$
$$= -(M_j \partial_i + \partial_j M_i) v^j - \hat\rho\, \partial_i \left(\frac{Nc^2 w}{u^\perp}\right)$$
$$+ \frac{N\hat\rho T_{\mathrm{f}}}{u^\perp} \eta_{,i}. \qquad (4.60)$$

In (4.60), substituting $M_i = \hat\rho s_i$ [according to (4.26b)] and using (4.59a) yields

$$\partial_t s_i = -v^j s_{i,j} - s_j v^j_{,i}$$
$$- \left(\frac{Nc^2 w}{u^\perp}\right)_{,i} + \frac{NT_{\mathrm{f}}}{u^\perp} \eta_{,i}, \qquad (4.61)$$

which has the geometric interpretation,

$$\partial_t(s_i\, \mathrm{d}x^i) = -\mathscr{L}_v(s_i\, \mathrm{d}x^i)$$
$$- \mathrm{d}\left(\frac{Nc^2 w}{u^\perp}\right) + \frac{NT_{\mathrm{f}}}{u^\perp}\, \mathrm{d}\eta. \qquad (4.62)$$

Here d denotes the exterior derivative tangential to the spacelike hypersurface and $\mathscr{L}_v$ is the Lie derivative with respect to the spatial components of the space–time velocity vector, cu^μ/u^0. Likewise, eqs. (4.58a, b) have similar geometrical inter-

pretations, namely,

$$\partial_t(\hat\rho\, \mathrm{d}^n x) = -\mathscr{L}_v(\hat\rho\, \mathrm{d}^n x), \qquad (4.63a)$$

$$\delta_t \eta = -\mathscr{L}_v \eta. \qquad (4.63b)$$

In Remark B, at the end of this section, we will show how eqs. (4.62–3) lead to additional conservation laws for gravitating fluids.

To show that eq. (4.61) implies the motion eq. (4.25c), we note that by (2.3) we have

$$-(c^2 w)_{,i} + T_{\mathrm{f}} \eta_{,i} = -\frac{1}{\rho_{\mathrm{f}}} p_{t,i}, \qquad (4.64)$$

and by (4.30) we have

$$\left(\frac{N}{u^\perp}\right)_{,i} = (u^\perp)^{-1} N_{,i}$$
$$+ N\left(-\frac{u^\perp}{2c^2}\right)\left(2t_j t^j_{,i} + t^j t^k g_{jk,i}\right), \qquad (4.65)$$

so that (4.61) becomes

$$\partial_t s_i = -v^j s_{i,j} - \frac{N}{u^\perp \rho_{\mathrm{f}}} p_{t,i} - \frac{c^2 w}{u^\perp} N_{,i}$$
$$- s_j v^j_{,i} + N s_j t^j_{,i} + \frac{N}{2} s^j t^k g_{jk,i}, \qquad (4.66)$$

where we have used (4.36d) in the form $s_k = u^\perp w t_k$. Now the fourth and fifth terms in (4.66) can be written as

$$s_j(Nt^j_{,i} - v^j_{,i}) \quad [\text{by (4.28b)}]$$
$$= s_j(cN^j_{,i} - t^j N_{,i}) \quad [\text{by (4.36d)}]$$
$$= cs_j N^j_{,i} - u^\perp w|t|^2 N_{,i} \quad [\text{by (4.31)}]$$
$$= cs_j N^j_{,i} - u^\perp w c^2 N_{,i} + \frac{wc^2}{u^\perp} N_{,i}.$$

Consequently, the third term in (4.66) cancels, leaving

$$\partial_t s_i = -v^j s_{i,j} - \frac{N}{u^\perp \rho_{\mathrm{f}}} p_{t,i} + cs_j N^j_{,i}$$
$$- u^\perp w c^2 N_{,i} + \frac{N}{2} s^j t^k g_{jk,i}. \qquad (4.67)$$

A short calculation similar to (2.28–30) gives

$$-v^j s_{i,j} + \frac{N}{2} s^j t^k g_{jk,i} = -Nt^j s_{i|j} + cN^j s_{i,j},$$

so that (4.67) becomes

$$\partial_t s_i = -Nt^j s_{i|j} - \frac{N}{u^\perp \rho_f} p_{f,i}$$
$$- c^2 w u^\perp N_{,i} + \left(cN^j s_j\right)_{,i} + cN^j (s_{i,j} - s_{j,i}).$$

(4.68)

Because the torsion is zero, we have $s_{i,j} - s_{j,i} = s_{i|j} - s_{j|i}$, so that (4.68) is equivalent to

$$\partial_t s_i = -\left(Nt^j - cN^j\right) s_{i|j} - \frac{N}{u^\perp \rho_f} p_{f,i}$$
$$- c^2 w u^\perp N_{,i} + \left(cs_j N^j\right)_{,i} - cN^j s_{j|i}$$
$$= -v^j s_{i|j} - \frac{N}{\rho_f u^\perp} p_{f,i} - c^2 w u^\perp N_{,i} + cs_j N^j_{|i},$$

(4.69)

which reproduces the motion eq. (4.25c), upon substituting $N(\rho_f u)^{-1} = N\sqrt{g}/\hat{\rho}$, by (4.26c).

Thus, the Lie–Poisson bracket (2.18) and Hamiltonian (4.51) yield the general relativistic adiabatic fluid eqs. (4.25a–c) directly in terms of the Eulerian physical variables $\{\hat{\rho}, \eta, M_i\}$. Because the map (4.50) preserves Poisson brackets, the equations in the canonically conjugate variables (P_A, q^A) also imply equations (4.25a–c). The fluid equations of motion in terms of canonically conjugate variables (P_A, q^A) are discussed in KN.

Finally, substituting variational derivatives (4.52d, e) and using the canonical Poisson bracket in the field variables (g_{ij}, π^{ij}) with the total Hamiltonian (4.46) leads to the ADM evolution equations, including matter (see, e.g., MTW (21.114–5), p. 525 and KN (2.31–4), p. 1012)

$$\partial_t g_{ij} = \frac{N}{\sqrt{g}}\left(2\pi_{ij} - \pi^m_m g_{ij}\right) + N_{i|j} + N_{j|i},$$

(4.70)

$$c^{-2}\partial_t \pi^{ij} = -\sqrt{g}\left[N\left(R^{ij} - \tfrac{1}{2}g^{ij}R\right) - N^{|ij} + N^{|m}_{|m}g^{ij}\right]$$
$$+ \frac{1}{\sqrt{g}}Nc^{-2}\left[\pi^{ij}\pi^m_m - 2\pi^i_m\pi^{mj}\right]$$
$$+ \tfrac{1}{2}g^{ij}\left(\pi^{lm}\pi_{lm} - \tfrac{1}{2}(\pi^m_m)^2\right)\Big]$$
$$- c^{-2}\left[N^i_{|m}\pi^{mj} + N^j_{|m}\pi^{mi} - \left(\pi^{ij}N^m\right)_{|m}\right]$$
$$+ \tfrac{1}{2}kc^{-4}\sqrt{g}\,NT^{ij}_{\|\|}.$$

(4.71)

Eq. (4.71) is the $\|, \|$ part of Einstein's equations

$$G^{ij}_{\|\|} = \frac{k}{2c^4}T^{ij}_{\|\|}.$$

(4.72)

It remains to prove that the $\perp\perp$ and $\perp\|$ components of Einstein's equations are preserved by the dynamics of the $\|, \|$ eqs. (4.72) and can, thus, be considered as nondynamical; that is, as initial-value constraints. An initial-value constraint for a given system is a relation or condition which, if imposed at a certain initial time, will be preserved under the dynamics of the system. Initial-value constraints are common in fluid dynamics. For example, in magnetohydrodynamics, the condition that the magnetic field be divergenceless is preserved by the equations of motion, if it holds initially. Likewise, the $\perp\perp$ and $\perp\|$ components of Einstein's equations are preserved under the $\|, \|$ dynamics and, thus, are initial-value constraints, by virtue of the Bianchi identities. This is shown explicitly in appendix B.

Consequently, we can summarize the results of this section, as follows: *The Hamiltonian structure for GRAF in the space of dynamical variables* $\{\hat{\rho}, \eta, M_i, g_{ij}, \pi^{ij}\}$, $i, j = 1, 2, \ldots, n$, *consists of the Hamiltonian functional* (4.46) *and a Poisson bracket which is the direct sum of the Lie–Poisson bracket* (2.18) *for the fluid variables* $\{\hat{\rho}, \eta, M_i\}$ [*defined in* (4.26c, d) *and* (4.36)] *and a canonical Poisson bracket for the field variables* (g_{ij}, π^{ij}).

Remarks. A) The present derivation via the map (4.50) of the Lie–Poisson bracket (2.18) for general relativistic fluids provides a constructive proof and confirmation of its earlier empirical discovery in BMW. The Lie–Poisson bracket for general relativistic fluids has the same form as for nonrelativistic fluids. From the present viewpoint, form-invariance of the Lie–Poisson bracket follows from form-invariance of the map from canonically conjugate Lagrangian fields to noncanonical, but physical, fluid variables [compare (4.50) with (2.4a, b) and (2.7)], with both sets of variables expressed in Eulerian coordinates.

B) Having shown the equivalence between the Hamiltonian eq. (4.60) and the motion eq. (4.25c),

we can use its Lie-derivative form (4.62) in combination with (4.63a, b) to find immediately that, with $s = s_i \, dx^i$, $i = 1, 2, \ldots, n$, in $(n+1)$ dimensions,

$$(\partial_t + \mathscr{L}_v)(ds \wedge d\eta) = 0, \tag{4.74}$$

so that, by (4.63a), in $(3+1)$ dimensions,

$$(\partial_t + \mathscr{L}_v)\Omega = 0, \quad \Omega := \hat{\rho}^{-1} \operatorname{curl} s \cdot \nabla \eta,$$

$$s_i := M_i / \hat{\rho} = (c\hat{\rho})^{-1} N \sqrt{g} \, T_i^0. \tag{4.75}$$

Thus, we have found the general-relativistic fluid conservation law $\partial_t C = 0$, with

$$C = \int d^3x \, \hat{\rho} \, \Phi(\eta, \Omega), \tag{4.76}$$

for an arbitrary function Φ of the indicated variables. The conserved functional C in (4.76) is in the kernel of the Poisson bracket (2.18); that is,

$$\{C, F\} = 0, \quad \forall F, \tag{4.77}$$

in the space of dynamical variables $\{\hat{\rho}, \eta, M_i, g_{ij}, \pi^{ij}\}$. Thus, the quantity C in (4.76) is a Casimir, conserved *independently* of the choice of Hamiltonian (4.51). The quantity Ω in (4.75) is the analog for GRAF of potential vorticity (Ertel's invariant) in geophysical fluid dynamics [cf. (2.38) and (3.44)]. The use of the Casimirs for semidirect product Lie–Poisson brackets in the study of Lyapunov stability of equilibrium states is discussed in section 5.

C) There is a well-known argument for treating the lapse and shift as Lagrange multipliers in the action (4.41). However, we avoid this argument and its associated complications ensuing because the Lagrangian density must then be regarded as being degenerate. For the Hamiltonian formalism, it is enough to show that the $\perp \perp$ and $\perp \|$ components of Einstein's equations are initial-value constraints; so that $\partial_t H = 0$, even for time-dependent lapse, shift, and metric, provided the initial data is appropriately constrained.

5. Comments on Casimirs and Lyapunov stability

This paper has focused on the common features of the Hamiltonian structures of NRAF, SRAF, and GRAF. The main result of this unified treatment is that in all three theories the fluid variables share the same Lie–Poisson bracket, when expressed in terms of the appropriate spaces of physical variables constructed here. As discussed in the introduction, one of our motivations for presenting this work as explicitly as possible is to facilitate "technology transfer", i.e., so that recently developed Hamiltonian techniques in both gravitation and fluid dynamics can cross-fertilize, particularly in the study of dynamic Lyapunov stability.

Recently there has been considerable development in the theory of Lie–Poisson structures for nonrelativistic and special-relativistic theories of fluids and plasmas, including magnetohydrodynamics, electromagnetic plasmas, Yang–Mills plasmas, and superfluids. For surveys of the nonrelativistic results, see, e.g., Holm, Kupershmidt and Levermore [15], Marsden et al. [16], and Marsden, Ratiu and Weinstein [14, 17]. For descriptions of special relativistic Hamiltonian structures, see Iwinski and Turski [19], and Holm and Kupershmidt [55, 57, 59, 60]. The Lie–Poisson brackets for these theories have been classified mathematically and their properties (e.g., nontrivial kernel) have been used to establish sufficient conditions for Lyapunov stability of equilibrium states for various fluid theories; see, e.g., Abarbanel et al. [49, 50], Arnold [51, 52], Hazeltine et al. [53], Holm and Kupershmidt [55, 57], Holm, Marsden and Ratiu [61], Holm, Marsden, Ratiu and Weinstein [13, 54], and Similon, Kaufman and Holm [18].

Finding the same Lie–Poisson structure among NRAF, SRAF, and GRAF suggests that the Lyapunov stability method, which relies so heavily on the conservation laws in the kernel of the Lie–Poisson bracket, should be applicable in all three cases, including gravitation. Indeed, the full Poisson bracket for GRAF, including the canonically conjugate gravitational fields, is formally

identical to the bracket for electrically charged fluids in the (E, A) representation, where E is the electric field, which is canonically conjugate to A, the magnetic vector potential. For such a charged fluid, Lyapunov stability conditions are known, in both the nonrelativistic case (Holm [62]) and the special relativistic case (Holm and Kupershmidt [55, 57]). The analogy bodes well for using the Hamiltonian approach to study Lyapunov stability for GR fluid theories, too.

A first step in this direction is to notice that certain (nondegenerate) equilibrium GRAF flows can be associated to critical points of the sum of the Hamiltonian H in (4.46) and the Casimirs C in (4.76) with an appropriate choice of the function Φ. Consider the following:

Proposition. With H and C defined in (4.46) *and* (4.76), *respectively, critical points of the sum $H + C$ are equilibrium states of GRAF dynamics.*

To understand this proposition, we first digress to discuss Casimirs further. Recall that the Casimirs have vanishing Poisson bracket (i.e., they "Poisson-commute") with any functional of the dynamical variables in the set $\{g_{ij}, \pi^{ij}, \hat{\rho}, \eta, M_i\}$. Thus, Casimirs are conservation laws, since they Poisson-commute with the Hamiltonian; but they are only kinematic, since their conservation is independent of the choice of the Hamiltonian, H, which generates the dynamics in this space of variables, under the rule for the Hamiltonian formalism,

$$\partial_t F = \{ H, F \}. \tag{5.1}$$

Here, $\{ , \}$ is the direct sum of the Lie–Poisson bracket for the fluid variables and the symplectic (canonical) Poisson bracket for the gravitational variables, and F is any functional of the dynamical variables. The Casimirs form an infinite family of these kinematic conservation laws, since they contain arbitrary (integrable) functions in their definition (4.76).

There are several explanations of how the Casimirs arise. From the viewpoint of the present

work, they appear via Noether's theorem. The action (4.24) in the Lagrangian configuration space admits the following symmetry transformation: relabel the Lagrangian variables so as to preserve the values of the physical fluid variables. A standard computation using Noether's theorem yields the conservation of C by virtue of the Lagrangian relabelling symmetry of the action (4.24). This symmetry transformation also plays a role in Lagrangian stability theory in astrophysics: Lagrangian variations that preserve the value of Ω in (4.75) are "nontrivial perturbations" in the sense of Friedman and Schutz [47] and Friedman [63], in Lagrangian perturbation theory.

Now, as for the proposition. By definition, a Casimir C satisfies

$$\{ C, F \} = 0, \quad \forall F.$$

So C generates no dynamics. In particular, the sum $H_C := H + C$ generates the same dynamics as the Hamiltonian H does alone. The critical states of H_C are equilibrium states of the dynamics, since the Poisson bracket $\{ H_C, F \} = \{ H + C, F \}$ vanishes when the first variation of H_C vanishes, i.e., when

$$0 = \delta H_C := DH_C\left(g_{ij}, \pi^{ij}, \hat{\rho}, \eta, M_i \right)$$

$$\cdot \left(\delta g_{ij}, \delta \pi^{ij}, \delta \hat{\rho}, \delta \eta, \delta M_i \right),$$

for arbitrary variations $(\delta g_{ij}, \delta \pi^{ij}, \delta \hat{\rho}, \delta \eta, \delta M_i)$. This observation proves the proposition above.

Thus, the existence of Casimirs (4.76) with their freedom in the choice of the function associates classes of equilibrium states with the critical states of certain functionals, H_C. The Lyapunov stability of these equilibria can then be studied by establishing whether there exist sufficient conditions, imposable on an equilibrium state (i.e., the "stability conditions"), under which we may obtain one or both of the following two situations. First, suppose $\delta^2 H_C$ is definite, i.e., the second variation

of H_C evaluated at the equilibrium state is definite in sign, under certain conditions on the equilibrium state. This situation implies linearized Lyapunov stability in terms of the norm given by $\delta^2 H_C$, which is conserved by the linearized dynamics (since, for Lie–Poisson systems, $\frac{1}{2}\delta^2 H_C$ is the Hamiltonian for the linearized dynamics, see Abarbanel et al. [50]). Second, suppose the conserved functional H_C is convex, i.e., the variation of H_C from its equilibrium value is bounded above and below by positive-definite quadratic forms; so that bounding norms exist for H_C. This situation implies Lyapunov stability for *finite* amplitude perturbations, in terms of the bounding norms. For a general description of Lyapunov analysis by this method as applied to nonrelativistic ideal fluid and plasma equilibria, see Holm, Marsden, Ratiu and Weinstein [13].

Whether sufficient conditions establishable by this method exist for Lyapunov stability of equilibrium states of GRAF remains an open question at this point. However, in view of the results of Friedman and Schutz [47] and Friedman [63] in this matter (namely, that for *rotating* GRAF equilibria sufficiently high wavenumber Lagrangian perturbations are always unstable) one should expect at most a conditional result; say, one that bounds the range of wavenumbers for which Eulerian variations are stable, in terms of equilibrium-state quantities. The linearization of the map (3.10) provides the relations needed for translating between the statements of Eulerian and Lagrangian infinitesimal perturbation theory for general relativistic fluids.

Acknowledgements

Appendix A

Derivation of the ADM motion equation for GRAF from its covariant form

The motion equation (4.25c) for a general relativistic adiabatic fluid is

$$\partial_t s_i = -v^j s_{i|j} - \frac{N\sqrt{g}}{\hat{\rho}} p_{t,i} - c^2 w u^{\perp} N_{,i} + c s_k N^k_{|i},$$

(A.1)

where $|$ denotes covariant differentiation in the hypersurface with lapse N and shift N^k, and components are taken with respect to a coordinate basis in the hypersurface i.e., in so-called adapted coordinates. Eq. (A.1) is derived as follows from the adapted-coordinates decomposition of the equation

$$\Box_\nu T^\nu_\mu := T^\nu_{\mu;\nu} = 0,$$

(A.2)

upon choosing μ in the direction i tangential to the hypersurface. Here the space–time covariant eq. (A.2) is decomposed into quantities that are covariant with respect to the N, N^k spatial hypersurface for a given slicing of space–time, following the description of adapted coordinates given in Isenberg and Nester [4].

In adapted coordinates which use the surface-compatible basis of vector fields $\{e_\perp, e_a\}$ along with its dual basis of one-form fields $\{\theta^\perp, \theta^a\}$, the space–time metric becomes

$$g = -\theta^\perp \otimes \theta^\perp + g_{ab}\theta^a \otimes \theta^b,$$

(A.3)

where g_{ab} is the induced metric in the hypersurface. Similarly, the inverse metric can be written

$$g^{-1} = -e_\perp \otimes e_\perp + g^{ab}e_a \otimes e_b.$$

(A.4)

(The symbol $e_\perp$ for a particular basis is called $n^\alpha \partial_\alpha$ in eq. (4.9′) of the text. Space–time quantities are written in bold-faced notation in the

appendices.) Applying these formulas for the contraction of the space–time fluid velocity vector, for example, leads to (4.31), i.e.,

$$-1 = g_{\alpha\beta} u^\alpha u^\beta$$
$$= -u^\perp u^\perp + g_{ij} u_\parallel^i u_\parallel^j. \tag{A.5}$$

Covariant $(n + 1)$ derivatives project into adapted coordinates by using the connection formulae (cf. Isenberg and Nester [4])

$$\Box_\perp \theta^\perp = -a_b \theta^b,$$
$$\Box_b \theta^\perp = K_{bc} \theta^c,$$
$$\Box_\perp \theta^b = -a^b \theta^\perp + \left[K_c^b - \tau_c^b - \theta^b (\mathcal{L}_{e_\perp} e_c) \right] \theta^c, \tag{A.6}$$
$$\Box_b \theta^a = K_b^a \theta^\perp - \Gamma_{cb}^a \theta^b.$$

Here $a_b \theta^b$ is a spatial 1-form, the "acceleration" of θ, defined in space–time as $a_\alpha := n_\alpha^{;\beta} n_\beta$ and obeying $a_\alpha n^\alpha = 0$ by virtue of $n_\alpha n^\alpha = -1$, so that $a^\perp := -a_\perp = 0$ (i.e., $a_\alpha = (0, a_b)$). The extrinsic curvature $K_{\alpha\beta} := -n_{\alpha;\beta}$ is also purely spatial, having only tangential components, since $n^\alpha K_{\alpha\beta} = 0 = K_{\alpha\beta} n^\beta$, again by $n_\alpha n^\alpha = -1$. The quantity in (A.6) $\mathcal{L}_{e_\perp} e_c = [e_\perp, e_b]$ is the Lie derivative vector field, and $\tau_b^a = Q^a_{\perp b}$ is a spatial tensor field, where $Q^\alpha_{\beta\gamma}$ is the torsion

$$Q^\alpha_{\beta\gamma} := \theta^\alpha \left(\Box_{e_\beta} e_\gamma - \Box_{e_\gamma} e_\beta - [e_\beta, e_\gamma] \right). \tag{A.7}$$

Vanishing of the torsion leads via the connection formulae (A.6) and their duals [obtained from covariant differentiation of $\theta^\alpha(e_\beta) = \delta_\beta^\alpha$] to the conditions

$$0 = Q^\perp_{\perp b} = a_b - e_b(\log N),$$
$$0 = Q^\perp_{ab} = K_{ba} - K_{ab},$$
$$0 = Q^m_{\perp b} = \tau_b^m, \tag{A.8}$$
$$0 = Q^m_{ab} = \theta^m \left(\nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a, e_b] \right) =: Q^m_{ab}.$$

The conditions (A.8) eliminate τ_b^m, Q^m_{ab}, and the antisymmetric part of K_{ab}. By (4.19) the symmet-

ric part of K_{ab} is

$$K_{ab} = -\tfrac{1}{2} \mathcal{L}_{e_\perp} g_{ab}. \tag{A.9}$$

Upon choosing a coordinate basis, the torsion-free conditions (A.8) give

$$a_b = N^{-1} N_{,b} \tag{A.10a}$$

and the Lie derivative in (A.6) becomes

$$\theta^a (\mathcal{L}_{e_\perp} e_b) = N^{-1} N^a_{,b}, \tag{A.10b}$$

in terms of the ADM lapse and shift. The ADM metric decomposition variables correspond to those here via the coordinate expressions

$$g = -\theta^\perp \otimes \theta^\perp + g_{ab} \theta^a \otimes \theta^b$$
$$= -N^2 c^2 \, dt \otimes dt + g_{ab}(dx^a + cN^a dt)$$
$$\otimes (dx^b + cN^b dt), \tag{A.11}$$
$$g^{-1} = -e_\perp \otimes e_\perp + g^{ab} e_a \otimes e_b$$
$$= -\frac{1}{N} \left(\frac{\partial}{c\partial t} - N^a \frac{\partial}{\partial x^a} \right) \otimes \frac{1}{N} \left(\frac{\partial}{c\partial t} - N^b \frac{\partial}{\partial x^b} \right)$$
$$+ g^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}. \tag{A.12}$$

The motion equation (A.1) is now to be written as the ith tangential component of the conservation law (A.2)

$$0 = T^\nu_{i;\nu} \tag{A.13}$$

expressed in terms of $s_i = (\hat\rho c)^{-1} \sqrt{g} \, T_i^\perp$. Using (4.7) leads to

$$0 = T^\nu_{i;\nu} = \left(\rho_f c^2 w u_i u^\nu \right)_{;\nu} + p_{f,i}$$
$$= \rho_f u^\nu \left(c^2 w u_i \right)_{;\nu} + p_{f,i}, \tag{A.14}$$

where $u_i = u_i^\parallel = s_i / cw$, so that the motion equation (A.14) becomes

$$0 = \rho_f u^\nu c s_{i;\nu} + p_{f,i}. \tag{A.15}$$

Using adapted-frame coordinates, we have

$$s := s_i \theta^i,$$
$$\Box_u s = u^\perp \Box_\perp (s_i \theta^i) + u_\parallel^k \Box_k (s_i \theta^i). \qquad (A.16)$$

Expanding (A.16) using the connection formulae (A.6) and the relations (A.10a, b) gives

$$\Box_u s = s_i \left[-u^\perp a^i + u_\parallel^k K_k^i \right] \theta^\perp$$
$$+ \left[u^\perp s_k (K_i^k - N^{-1} N_{,i}^k) + u^\perp e_\perp (s_i) \right.$$
$$\left. + u_\parallel^k s_{i|k} \right] \theta^i, \qquad (A.17)$$

where $e_\perp (s_i) := N^{-1}(\partial_{ct} - N^k \partial_k) s_i$. Note that $u_\alpha u^\alpha = -1$ implies $\Box_u u = 0$, so the $\theta^\perp$ component of (A.17) yields the relation

$$u^\perp a_i = u_j^\parallel K_i^j, \qquad (A.18)$$

which will be useful in a moment. Taking the θ^i component of (A.17) we have

$$\Box_u s_i := u^\nu \Box_\nu s_i$$
$$= u^\perp \left[e_\perp (s_i) + s^j K_{ji} - N^{-1} s_j N_{,i}^j \right.$$
$$\left. + (u^\perp)^{-1} u_\parallel^k s_{i|k} \right]$$
$$= u^\perp N^{-1} \left[s_{i,ct} - N^k s_{i,k} + N s^j K_{ji} - s_j N_{,i}^j \right.$$
$$\left. + \frac{N}{c} t^k s_{i|k} \right], \qquad (A.19)$$

where the definition (4.27a) for t^k (namely, $t^k/c = u_\parallel^k / u^\perp$) has been used. Relation (A.18) now becomes, by (A.10a) and recalling $s_j := cwu_j^\parallel$ from (4.36),

$$cwu^\perp N_{,i} \quad \text{[by (A.10a)]} = cwu^\perp Na_i \quad \text{[by (A.18)]}$$
$$= Ncwu_j^\parallel K_i^j \quad \text{[by (4.36)]} = Ns_j K_i^j. \qquad (A.20)$$

The covariant derivative (A.19) becomes, after

substitution of (A.20) for the term $Ns^j K_{ji}$,

$$\Box_u s_i = u^\nu s_{i;\nu}$$
$$= u^\perp N^{-1} \left[s_{i,ct} - N^k s_{i,k} + cwu^\perp N_{,i} - s_j N_{,i}^j \right.$$
$$\left. + \frac{N}{c} t^k s_{i|k} \right] \quad \text{[by (4.27b)]}$$
$$= u^\perp N^{-1} \left[s_{i,ct} + \frac{v^k}{c} s_{i|k} + N^k s_{i|k} - N^k s_{i,k} \right.$$
$$\left. - s_k N_{,i}^k + cwu^\perp N_{,i} \right]$$
$$= u^\perp N^{-1} \left[s_{i,ct} + \frac{v^k}{c} s_{i|k} - s_k N_{|i}^k + cwu^\perp N_{,i} \right]. \qquad (A.21)$$

Substituting the last expression for $u^\nu s_{i;\nu}$ into (A.15), we obtain

$$0 = \rho_f u^\nu c s_{i;\nu} + p_{f,i}$$
$$= \rho_f u^\perp N^{-1} \left[s_{i,t} + v^k s_{i|k} - cs_k N_{|i}^k + c^2 wu^\perp N_{,i} \right]$$
$$+ p_{f,i}. \qquad (A.22)$$

Upon dividing (A.22) through by $\rho_f u^\perp N^{-1}$, which by (4.26c) equals $\hat{\rho}/(N\sqrt{g})$, we arrive at an equation equivalent to (A.13), but in the desired form (A.1).

Appendix B

Preservation of the initial-value constraints

An initial-value constraint for a given system is a relation which, if true at a certain initial time, is preserved under the dynamics of the system. In this appendix, we verify that the $\perp\perp$ and $\perp\parallel$ components of Einstein's equations (4.1a) are initial-value constraints for the $\parallel, \parallel$ components, by virtue of the Bianchi identities for the entire system of Einstein's equations. Let us decompose Einstein's equations into perpendicular and paral-

lel components (cf. (4.21a–c)

$$C^{\perp}{}_{\perp} := G^{\perp}{}_{\perp} - \frac{k}{2c^4} T^{\perp}{}_{\perp} = \frac{1}{2\sqrt{g}} \mathcal{H}, \qquad (B.1a)$$

$$C^{\perp\parallel}{}_{i} := G^{\perp\parallel}{}_{i} - \frac{k}{2c^4} T^{\perp\parallel}{}_{i} = \frac{1}{2\sqrt{g}\,c^2} \mathcal{J}_i, \qquad (B.1b)$$

$$C^{i\parallel}_{\parallel j} := G^{i\parallel}_{\parallel j} - \frac{k}{2c^4} T^{i\parallel}_{\parallel j}. \qquad (B.1c)$$

The Bianchi identities are expressed here as

$$\square_\nu \left(G^\nu_\mu - \frac{k}{2c^4} T^\nu_\mu \right) = 0. \qquad (B.2)$$

In particular, the projected Bianchi identities are

$$0 = \square_\nu C^{\perp\nu} = g^{\perp\perp}\square_\perp C^{\perp}{}_{\perp} + g^{im}\square_i C^{\perp\parallel}_m, \quad (B.3a)$$

$$0 = \square_\nu C^{\parallel\nu}_j = \square_\perp C^{\perp\parallel}_j + \square_k C^{k\parallel}_{\parallel j}. \qquad (B.3b)$$

From these identities, we calculate the evolutionary equations

$$\partial_{ct} C^{\perp}{}_{\perp} = N^m C^{\perp}{}_{\perp,m} + N C^{\perp}{}_{\perp} K^m_m + \left(N C^{\perp\parallel}_m \right)^{|m}, \qquad (B.4a)$$

$$\partial_{ct} C^{\perp\parallel}_j = N^m C^{\perp\parallel}_{j,m} + N C^{\perp\parallel}_j K^m_m - \left(N C^{\parallel m}_{j\parallel} \right)_{|m}. \qquad (B.4b)$$

Therefore, provided $C^{m\parallel}_{\parallel j}$ vanishes, i.e., provided the $\parallel, \parallel$ components of Einstein's equations are satisfied, the values $C^{\perp}{}_{\perp} = 0 = C^{\perp\parallel}_j$ will be preserved by virtue of the Bianchi identities (B.3a, b). Equivalently, the values $\mathcal{H} = 0 = \mathcal{J}_i$ will be preserved. The rest of this appendix is devoted to the proof of eqs. (B.4a, b).

To prove relation (B.4a), we calculate in adapted coordinates (as in appendix A, following Isenberg and Nester [4]) the quantity in (B.3a),

$$\square_\nu C^{\perp\nu} = 0. \qquad (B.5)$$

Let $\{\theta^\perp, \theta^i\}$ be the dual basis in appendix A and

$$C^\perp := C^{\perp}{}_{\perp} \theta^\perp + C^{\perp\parallel}_i \theta^i. \qquad (B.6)$$

Then, using the connection formulae (A.6) we have

$$\square_\perp C^\perp = \left[e_\perp \left(C^{\perp}{}_{\perp} \right) - C^{\perp\parallel}_i a^i \right] \theta^\perp$$
$$+ \left[e_\perp \left(C^{\perp\parallel}_j \right) - C^{\perp}{}_{\perp} a_j \right.$$
$$\left. + C^{\perp\parallel}_i \left(K^i_j - \theta^i (\mathscr{L}_{e_\perp} e_j) \right) \right] \theta^j, \qquad (B.7a)$$

$$\square_i C^\perp = \left[e_i \left(C^{\perp}{}_{\perp} \right) + C^{\perp\parallel}_j K^j_i \right] \theta^\perp$$
$$+ \left[C^{\perp}{}_{\perp} K_{ij} + e_i \left(C^{\perp\parallel}_j \right) - C^{\perp\parallel}_k \Gamma^k_{ji} \right] \theta^j. \qquad (B.7b)$$

Taking $e_i = \partial_i$, the $\theta^\perp$ component of (B.7a) and the θ^j component of (B.7b) yield the Bianchi identity

$$0 = \square^\nu C_\nu^\perp = \square_\nu C^{\perp\nu} = g^{\perp\perp}\square_\perp C^{\perp}{}_{\perp} + g^{im}\square_i C^{\perp\parallel}_m$$
$$= -e_\perp \left(C^{\perp}{}_{\perp} \right) + C^{\perp\parallel}_i a^i + C^{\perp}{}_{\perp} K^m_m + \left(C^{\perp\parallel}_m \right)^{|m}. \qquad (B.8)$$

Rearranging (B.8) using $e_\perp = N^{-1}(\partial_{ct} - N^i \partial_i)$ and $a_i = N^{-1} N_{,i}$ gives

$$\partial_{ct} C^{\perp}{}_{\perp} = N^i C^{\perp}{}_{\perp,i} + N C^{\perp}{}_{\perp} K^i_i + \left(N C^{\perp\parallel}_i \right)^{|i}. \qquad (B.9)$$

This proves relation (B.4a).

A similar calculation involving $C^{\perp\parallel}_j$ is needed to prove (B.4b). The necessary Bianchi identity is (B.3b). Let

$$C_j := C^{\perp\parallel}_j e_\perp + C^{m\parallel}_{\parallel j} e_m. \qquad (B.10)$$

The basis $\{e_\perp, e_m\}$ satisfies $\theta^a(e_b) = \delta^a_b$ and $\theta^\perp(e_\perp) = 1$, so the connection formulas (A.6) revert to those in Isenberg and Nester [4]. Namely,

$$\square_\perp e_\perp = a^b e_b,$$

$$\square_b e_\perp = -K^a_b e_a,$$

$$\square_\perp e_b = a_b e_\perp + \left[-K^a_b + \tau^a_b + \theta^a (\mathscr{L}_{e_\perp} e_b) \right] e_a, \qquad (B.11)$$

$$\square_c e_b = -K_{bc} e_\perp + \Gamma^a_{bc} e_a.$$

Whence, for zero torsion conditions (A.8), we find

$$\Box_\perp C_j^{\parallel} = \left[e_\perp \left(C^{\perp\parallel} \right) + C_{\parallel j}^{k\parallel} a_k \right] e_\perp$$
$$+ \left[e_\perp \left(C_{\parallel j}^{m\parallel} \right) + C^{\perp\parallel}_j a^m \right.$$
$$\left. + C_{\parallel j}^{k\parallel} \left(-K_k^m + \theta^m (\mathscr{L}_{e_\perp} e_k) \right) \right] e_m,$$
(B.12a)

$$\Box_k C_j^{\parallel} = \left[C^{\perp\parallel}_{j,k} - C_{\parallel j}^{m\parallel} K_{mk} \right] e_\perp$$
$$+ \left[-C^{\perp\parallel}_j K_k^m + e_k \left(C_{\parallel j}^{m\parallel} \right) + C_{\parallel j}^{l\parallel} \Gamma_{lk}^m \right] e_m.$$
(B.12b)

Adding the $e_\perp$ component of (B.12a) to the summed e_k components of (B.12b) using the representations $e_\perp = N^{-1}(\partial_{ct} - N^i \partial_i)$, $e_k = \partial_k$, and the zero-torsion condition $a_k = N^{-1} N_{,k}$ gives the Bianchi identity (B.3b) as

$$0 = \Box_\nu C_j^{\nu\parallel} = \Box_\perp C^{\perp\parallel}_j + \Box_k C_{\parallel j}^{k\parallel}$$
$$= N^{-1} \left(\partial_{ct} - N^i \partial_i \right) C^{\perp\parallel}_j - C^{\perp\parallel}_j K_k^k$$
$$+ \left(C_{\parallel j}^{k\parallel} \right)^{|k} + N^{-1} N_{,k} C_{\parallel j}^{k\parallel}.$$
(B.13)

Rearranging (B.13) gives

$$\partial_{ct} C^{\perp\parallel}_j = N^i C^{\perp\parallel}_{j,i} + N C^{\perp\parallel}_j K_k^k - \left(N C_{\parallel j}^{k\parallel} \right)_{|k}.$$
(B.14)

This proves relation (B.4b).

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